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Abstract

The topic of risk aggregation arises from the need to incorporate in a single measure the overall exposure to the different risk types. In general, the methodologies adopted for the purposes of risk integration are based on the principle that the overall economic capital is lower than the simple algebraic sum of economic capital measures related to individual risks. This phenomenon, due to the existence of an imperfect correlation between the risks, determines, in line with portfolio theory, a "diversification benefit". The issue of risk allocation subsequently arises when the risk value of the diversified aggregated loss needs to be reassigned to the different risk classes. A similar issue has been solved in the framework of cooperative Game Theory, where the Shapley value provides a player-specific contribution of the total surplus generated by the coalition. The paper proposes a novel application of the Shapley formula in the ICAAP context (Pillar II - economic view). In particular, we show that the Shapley value is the unique solution to the allocation problem of an overall risk value, granting the fundamental requested properties, including the efficiency one. An exemplificative model application is reported, as well as a comparison with a benchmark methodology. The experimental part shows the advantages of the novel approach in terms of precision and reliability of the estimates. Finally, it is important to mention that the presented framework can be applied also in other contexts such as, for instance, in the risk class attribution of the operational risk.

Il tema dell'aggregazione dei rischi deriva dalla necessità di incorporare in un'unica misura l'esposizione complessiva ai diversi tipi di rischio. In generale, le metodologie adottate ai fini dell'integrazione dei rischi si basano sul principio che il capitale economico complessivo è inferiore alla semplice somma algebrica delle misure di capitale economico relative ai rischi individuali. Questo fenomeno, dovuto all'esistenza di una correlazione imperfetta tra i rischi, determina, in linea con la teoria di portafoglio, un "beneficio di diversificazione". La questione dell'allocazione del rischio sorge successivamente, quando il valore del rischio della perdita aggregata diversificata deve essere riassegnato alle diverse classi di rischio. Un problema simile è stato risolto nell'ambito della Teoria dei Giochi, dove il valore Shapley fornisce il contributo specifico del singolo giocatore al surplus totale generato dalla coalizione. Il presente articolo propone una nuova applicazione della formula di Shapley nel contesto dell'ICAAP (II pilastro - vista gestionale). In particolare, mostriamo che il valore Shapley è l'unica soluzione al problema di allocazione del valore di rischio complessivo che garantisce le proprietà fondamentali richieste, compresa quella dell'efficienza. Viene riportata un'applicazione modello, nonché un confronto con una metodologia di benchmark. La parte sperimentale mostra i vantaggi del nuovo approccio in termini di precisione e affidabilità delle stime. Infine, è importante ricordare che l'approccio presentato può essere applicato anche in altri contesti come, ad esempio, nell'attribuzione della classe di rischio del rischio operativo.

Key Words:

Shapley value, VaR, risk allocation, risk integration, diversification benefit, copula

1. Risk aggregation framework and the issue of risk allocation

A wide literature tackles the issue of risk aggregation and risk allocation from a statistical and mathematical point of view, starting from the general discussion in [1] and the derivation of principles of allocation by using properties of risk measures in [2] and [3].

As regards the risk aggregation, the simplest form is the mere sum of the economic capital related to each different type of risk; this method is called "building block" and implicitly assumes a perfect positive correlation between risks, therefore it is configured as the most prudential approach. The assumption that risks are perfectly correlated is, however, very unlikely, since the risks do not always translate into actual losses at the same time; the allocation of economic capital among different risk types, therefore, can lead to diversification benefits, entailing less overall capital absorbed. More advanced aggregation approach can be achieved through Var-CoVar methodology or rather copula functions, which allow to describe with greater accuracy the structure of dependence between risks that determines their joint movement as in [4], [5] and [6].
The copula approach can be used to describe any multivariate distribution as a set of marginal distributions and a copula function, which represents the pure dependency structure between the random variables regardless of the marginal distributions of the variables themselves. The possibility of modeling and simulating any multivariate distribution therefore allows a more accurate description of the dependence relationships between risks.

Given a model for the aggregated losses of a number *d* of risk classes, the capital allocation topic (see [7]) tackles the reallocation of the overall capital requirement into additive contributions attributable to the different risk classes (or sub-portfolios if we are considering an aggregated portfolio) that make up the overall risky events. In this context, in [8] [9] the specific framework of the Euler theorem for risk allocation is treated.

The innovative contribution of the present work is to employ the Shapley value in the context of ICAAP capital allocation. The Shapley value is a solution arisen in the framework of cooperative game theory, named in honor of Lloyd Shapley who won the Nobel Prize in Economics in 2012. The solution was originally introduced in [10] and well described in [11] and in [12].
It is important to mention that there are many other applications of the Shapley Value in economics: the systemic risk monitoring [13], the derivation of measures for banks' systemic importance [14], the allocation in a portfolio under a Markowitz mean-variance choice paradigm [15] and the allocation of Savings in a Supply Chain [16].

2. Mathematical setting: the Shapley value

The Shapley value is a typical solution arising in the framework of the game theory for the problem of assigning the fair reward to each member of a coalition following an overall gain (payoff).

The formal definition of a cooperative game of a set D of d players rely on the function called *characteristic function* that maps each subset of players $S \subseteq D$ to a real number corresponding to the payoff the members of S get by cooperating. In formula, the characteristic function is:

$$v: P(D) \rightarrow R$$

where $P(D)$ is the power set of D, i.e. the set of all the possible subset of D, with the condition that the function returns zero on the empty subset $v(\emptyset) = 0$.

In the following, we will uniquely identify each player D_i by the integer i in the range $\{1, \dots, d\}$, so that the characteristic function will be defined in the domain of the 2^d subsets of the range $\{1, \dots, d\}$:

$$v: P(\{1, \dots, d\}) \rightarrow R$$

Under quite natural assumptions, the solution of this problem is given by the Shapley Value, that is a function defined for each player i that gives the fair gain of the player i for the characteristic function v.
With the usual symbolic notation, the Shapley value is computed as:

$$\phi_i(v) = \sum_{S \subseteq D/\{i\}} \frac{|S|!(|D|-|S|-1)!}{|D|!} \left(v(S \cup \{i\}) - v(S)\right) \quad \textbf{Eq. 1}$$

where the sum is done on all subsets of D not including the player i.

Let us try to understand the meaning of Eq.1 for a chosen player i. Fixing a subset S with cardinality $|S| = k$, the term $v(S\cup\{i\})-v(S)$ represents the extra contribution that the player i adds to the coalition S.

We can compute the average of these contributes for all the subsets with a fixed cardinality k:

$$\sum_{S \subseteq D/\{i\}\ with\ |S|=k} \frac{v(S \cup \{i\}) - v(S)}{\dfrac{|D-1|!}{k!\,(|D|-1-k)!}}$$

Where $\frac{|D-1|!}{k!(|D|-1-k)!}$ is the number of the possible subsets with cardinality k, i.e. the number of combinations of $|D - 1|$ elements in groups with cardinality k.

Now we should compute the average of the introduced quantity for all the possible cardinalities of the subsets. These cardinalities are $K = \{0,1,2, \dots, |D| - 1\}$ that is a set with cardinality $|D|$.

This second average is computed as

$$\sum_{k \in K} \left(\sum_{S \subseteq D/\{i\}\ with\ |S|=k} \frac{v(S \cup \{i\}) - v(S)}{\dfrac{|D-1|!}{k!\,(|D|-1-k)!}} \right) \frac{1}{|D|} = \sum_{S \subseteq D/\{i\}} \frac{|S|!\,(|D|-|S|-1)!}{|D|!} \left(v(S \cup \{i\}) - v(S)\right) = \phi_i(v)$$

One fundamental property of this solution is the Efficiency property. In fact, the sum of the Shapley values of all players equals the value of the grand coalition:

$$\sum_{i \in D} \varphi_i(v) = v(D) \quad \textbf{Eq. 2}$$

The super-additive property grants the individual rationality, i.e. the fact that in any coalition a player receives more than what he could get on his own.

3. Shapley Capital Allocation methodology

In this section we introduce the mathematical framework of risk allocation and we present the novel methodology of Shapley capital allocation.

Let $L_1, \dots, L_d$ represent the loss random variables associated to the different risk classes and let

$$L = \sum_{i=1}^{d} L_i \quad \textbf{Eq. 3}$$

be the overall loss random variable having an aggregated risk measure $\rho(L)$, e.g. the $VaR_{99\%}(L)$.

The goal is to find a mapping that takes as input the random variable of individual losses $L_1, \dots, L_d$ and the aggregated risk measure and yields a vector of risk contributions (Allocated Capital) for each risk class $(AC_1, \dots, AC_d)$ that in general are different from the single risk measures $\rho(L_i)$.

What is strictly required is that the sum of the risk contributions gives the contribution of the aggregated losses (efficiency property), in formula:

$$\rho(L) = \sum_{i=1}^{d} AC_i \quad \textbf{Eq. 4}$$

The Shapley value solution introduced in the previous section can be used to compute the fair Allocated Capital for each risk class starting from the overall aggregate capital.

Let us define a coalition game where the set of players *D* is given by the *d* different risk classes; as explained in the previous section, we identify each player D_i by the integer *i* in the range (1,…, d); so $D = \{D_1, \dots, D_d\} = \{1, \dots, d\}$.

Once we fix a subset of integer $S = S_1, \dots, S_{|S|} \subseteq D$ (for instance *S* = {3,5,8}), the characteristic function *v*(*S*) can be defined by:

$$v(S) = \rho(L_S) \quad \textbf{Eq. 5}$$

where L_S represents the loss random variable of the aggregated losses of the subset *S:*

$$L_S = \sum_{j \in S} L_j$$

Basically, the characteristic function is given by the risk measure computed on the aggregated losses of a certain number of risk classes (a subset of the whole set). Suppose we are interested in the VaR, in practice, we have to compute the aggregated losses and the corresponding VaR for each possible subset of the risk classes.

Note that the efficiency property introduced above ensures that the sum of the risk values is equal to the overall loss.

4. Sanity check of Shapley Capital Allocation methodology for VaR and ES

In order to verify the correctness of the Shapley capital allocation approach, we carried out a simulated case study where the risk classes equally contribute to the overall risk, providing an analytical solution to the allocation problem. Then, in order to compare the results, we exploit a benchmark procedure based on the Window Conditional Expectation (WCE) [17] where for each risk, an average contribution is computed over the scenarios of the single losses in correspondence to the scenarios of the aggregated losses around (±5%) the risk measure. As a case study, we exploited 10 independent standard Gaussian random variables $L_i \sim N(0,1)$ and we simulated a sample of 100,000 10-dimensional loss vectors.

In this setting, we computed the overall capital given by the VaR, i.e. the percentile *p=99%* and we applied the two procedures for the reallocation of the capital into each risk class.

As the losses are independent, we expect that the marginal allocated capital is constant for the all the risk classes and given by:

$$AC_i = \frac{VaR_L(p)}{10} = \frac{\Phi^{-1}_{N(0,\sqrt{10})}(p)}{10}, \quad \textbf{Eq. 6}$$

where $\Phi^{-1}{}_{N(0,\sqrt{10})}$ is the percentile point function (inverse of CDF) of $L = \sum_{i=1}^{d} L_i = N(0, \sqrt{10})$. In the case of p=99%, the theoretical solution is given by $AC_i = 0.736$

In the following Figure we report for the percentile 99% the allocated capital for the 10 different classes (x-axis) for the two methodologies.

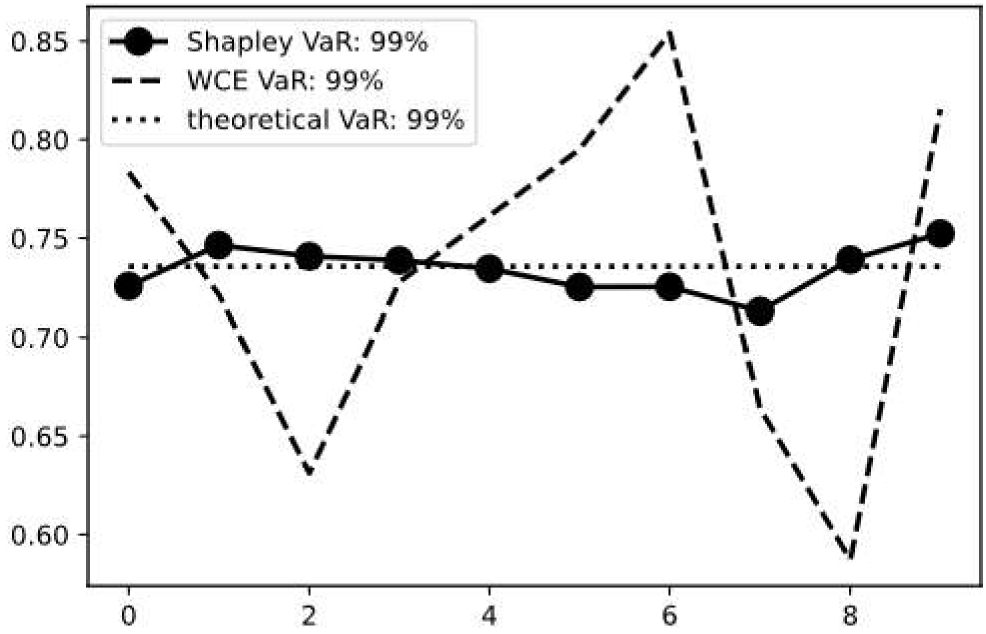

Figure 1: Allocated capital (VaR) into the 10 classes for 99 percentile

We observe that the Shapley methodology provides results closer to the theoretical one.

A similar exercise can be done for the Expected Shortfall (ES), i.e. the average loss in the worst p% cases. The theoretical value can be computed as:

$$AC_i = \frac{ES_L(p)}{10} = \frac{\sqrt{10}\,\varphi(\Phi^{-1}_{N(0,1)}(1-p))}{(1-p)\cdot 10} = 0.843 \quad \textbf{Eq. 7}$$

where φ is the pdf of the standard gaussian distribution.

In the following Figure we report for the ES 99% the allocated capital for the 10 different classes (x-axis).

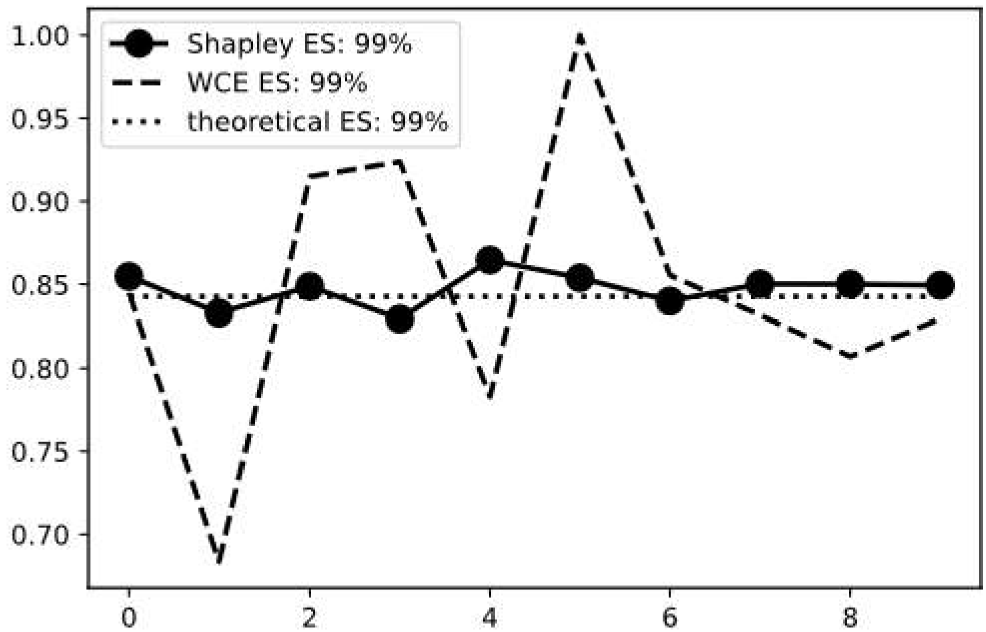

Figure 2: Allocated capital (ES) into the 10 classes for 99 percentile

Also in this case we observe that the Shapley methodology gives more accurate and stable results with respect to the benchmark methodology.

The standard deviation of the estimates are reported in the following table for both the Shapley and the WCE methodologies.

	Std (VaR)	Std (ES)
Shapley	0.011	0.01
WCE	0.081	0.082

Table 1

5. Real life case study

In order to present an illustrative application, let us introduce the approach where a risk integration framework can be calibrated on a Student t copula (see [18]), which is an ideal and flexible model choice in order to adequately describe the dependence structure between risks.

A survey on literature on this topic is reported in [7]. Then, we adopt the methodology based on Shapley value and a benchmark procedure for the allocation of the risk into the different classes.

The conceptual approach is divided into the following phases:

1. Define a joint distribution through the:
    a. assumption/identification of the marginal distributions related to each risk class via the time series representing the loss variables;
    b. assumption of the dependence structure between risks being represented by a copula function: in this case a t copula, defined by two parameters - v degree of freedom and R correlation matrix. By keeping the matrix R fixed to historical correlation, the estimate of v degrees of freedom can be carried out through maximum likelihood estimation.
2. Once the modeling of the copula has been completed, Monte Carlo simulations can be carried out on its realizations and, by means of the inverse distribution functions of the marginals (quantile functions), these realizations are translated into integrated economic capital measures.
3. Selection of the percentile on the sum of the losses, expressive of the measure of the diversified overall economic capital, in line with the determined VaR confidence level for individual risks.
4. The diversified overall economic capital hence needs to be marginally reallocated on the risk classes, in order to consistently attribute the contribution of each risk in the diversification process.

It should be highlighted how the hypothesis at step (1) is implicitly assumed also in the VaR-Covar aggregation methodology and how the hypothesis at step (1.b) is equivalent to generalize the Gaussian case implicit in this method, since for large v the t copula converges to the Gaussian one, while for small v it denotes strong non-linear dependencies.

Step (4) is the instance in which the subject of the present paper comes into play.

Let us envisage a structure composed by 6 risk classes, marginally distributed as Gaussian or Log-Normal random variables with arbitrary parameters, joined by a Student t Copula.

It is reasonable to consider distributions with zero expected value, since an expected loss should be managed directly through adequate P&L adjustments and not by economic capital.

It is also reasonable to evaluate some risk classes as pure Gaussian, assuming negative loss observations and thus potential profits, while others should be constricted only in the positive plan (no profits can be envisioned by definition).

Log-normal distribution could respond to this need.

Risk class	Marginal Distribution
Risk #1	$L_i \sim \text{logN}(0,2.5)$
Risk #2	$L_i \sim \text{N}(0,50)$
Risk #3	$L_i \sim \text{N}(0,500)$
Risk #4	$L_i \sim \text{N}(0,100)$
Risk #5	$L_i \sim \text{logN}(0,2)$
Risk #6	$L_i \sim \text{N}(0,75)$

Table 2

Let us also assume a Student t Copula with v=2 degree of freedom (strong tail dependence) and arbitrary semi-definite positive matrix R as follows:

Risk	Risk #1	Risk #2	Risk #3	Risk #4	Risk #5	Risk #6
Risk #1	100%	30%	70%	50%	20%	40%
Risk #2	30%	100%	60%	60%	70%	30%
Risk #3	70%	60%	100%	80%	50%	50%
Risk #4	50%	60%	80%	100%	60%	30%
Risk #5	20%	70%	50%	60%	100%	70%
Risk #6	40%	30%	50%	30%	70%	100%

Table 3

Running 5 million realizations of the copula and translating them into integrated economic capital measures with the associated inverse marginal distributions, we obtain about a 14% benefit on the overall economic capital measured with the building block approach (mere sum of the VaR at 99.9% level).

The following table sums up the results of the Shapely Capital Allocation versus the WCE benchmark methodology:

	Diversification benefit on individual risks		***Weight on diversification effect***	
Risk class	SHAPLEY	WCE	SHAPLEY	WCE
Risk #1	-12%	-7%	38%	23%
Risk #2	-30%	-44%	7%	10%
Risk #3	-9%	-10%	19%	21%
Risk #4	-19%	-24%	8%	11%
Risk #5	-27%	-32%	19%	22%
Risk #6	-29%	-39%	9%	13%

Table 4

Such evidence suggests that the choice of the reallocation methodology is a key passage in a consistent risk integration framework, in order to coherently attribute the diversification benefit to each risk in relation to its true contribution to the latter.

In particular, the authors note that the differences are higher for the most relevant risk classes, e.g. the Risk #1.

The differences emerged are significant especially when pursuing a risk-adjusted business model logic.

A more punctual assessment permits to estimate more accurately the consequent risk-adjusted measures (e.g. RORAC), leading not only to a valuable risk management but also to an effective planning of the economic resources distribution.

6. Conclusions

The topic of risk aggregation is a key concept in the framework of ICAAP (Pillar II - economic view). Then, the issue of risk allocation arises when the risk value of the diversified aggregated loss needs to be reassigned to the different risk classes. This problem is particularly important for a correct evaluation of the most relevant risks with respect to the Bank business.

In the present paper we successfully applied the Shapley Value for determining the correct contribution of a single risk class to the overall risk value. This innovative application is able to solve the allocation problem in a mathematically sound framework and in an effective way with respect to the other methodologies. The advantage of this approach with respect to others is shown in a simulation case study and in a real-life application example.

Future work can be done in order to apply the Shapley Value allocation to other risk management contexts. For instance, it is important to underline that the presented framework can be applied in the risk class attribution of the operational risk. The computational issues of Shapley value in the case of high dimensional set can be further explored by considering approximated solutions.

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