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Alternative Stochastic Binomial Trees for Quantitative Analysis of Convertible Bonds

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Abstract
The objective of the present study is to implement the alternative stochastic binomial trees for the evaluation and estimation of the main sensitivity measures of convertible bonds, thus filling a gap in scientific literature. The paper proposes the implementation of the Haahtela, Jarrow-Rudd and Tian numerical schemes and explores the characteristics, convergence properties and reliability of these evaluation tools. A comprehensive case-study considering the German market, which is an extremely active market in the issuance and trading of these hybrid instruments, is also illustrated.

Key Words: Convertible bonds, alternative stochastic binomial trees, Cox-Ross-Rubinstein, Haahtela, Jarrow-Rudd, Tian

JEL codes: C53, C63, G12, G32

1) Introduction
The valuation of many complex financial instruments, such as convertible bonds, typically requires numerical methods due to their option-like features.

A convertible bond is a hybrid security that retains most characteristics of straight debt while also offering the upside potential associated with underlying common stock. A convertible bond is a corporate security giving the bondholder the right, but not the obligation, to convert the bond into another security, typically the ordinary shares of the issuing company, under specific conditions. Once converted into ordinary shares, they cannot revert into bonds. Due to their structure and their option feature, convertibles show characteristics of both debt and equity instruments, leading to their classification as hybrid instruments. A similar instrument is convertible preferred stock, which means preference shares that can be converted into ordinary shares based on specific terms. Convertibles play an important role in corporate finance and have benefited from advanced valuation models originally created for option markets. Their hybrid nature initially presented challenges in analysis and valuation, but modern techniques have largely solved these issues. As a result, issue volumes increased steadily through the 1990s, particularly during rising stock markets. Convertible bonds are fixed-coupon securities issued with the option to convert into equity, at the bondholder's discretion, under pre-determined terms. These bonds are usually subordinated, and only companies with strong credit ratings can issue them. The market perception of the issuer's stock performance is also critical, since investors are purchasing the right to subscribe for shares at a future date, potentially at a premium over the market price. Consequently, the price of a convertible bond fluctuates following changes in the underlying stock price and in interest rates. Convertibles are typically medium to long-term instruments with maturities of 10 to 20 years, and their coupons are typically lower than those on non-convertible bonds from the same issuer. In addition to basic fixed-coupon convertibles, various other instruments are also available. They include zero-coupon convertibles, issued at a deep discount with a low probability of conversion, and discount convertibles. Some convertibles are callable by the issuer, allowing them to force conversion under certain conditions. They are called convertible calls and reduce the bondholder's discretion, potentially leading to unfavourable terms. Conversely, puttable convertibles allow bondholders to redeem the bond or convert it at their pleasure, providing downside protection. Premium put convertibles can only be converted on a single date, while rolling put convertibles offer multiple conversion dates and are generally issued with a lower coupon.

Another variant is the exchangeable security, a bond issued by one company that is convertible into the shares of another company in which the issuer holds a significant interest. Bonds with warrants are convertible bonds issued with an attached warrant that can be traded individually. Step-up convertibles and preference shares are newer innovations, offering a fixed coupon for the first few years before increasing the coupon until maturity or conversion.

Convertible bonds have a long history in capital markets, with the first issuances by U.S. utility companies in the 19th century. In 1997, the global convertibles market was valued at over $360 billion, with the U.S. as the largest issuer, historically dominated by utility and transport companies. Unlike domestic and international securities, convertibles are primarily exchange-traded, offering more liquidity and transparency compared to OTC bonds. However, liquidity still depends on the number of market makers and the volume of the issue, making some convertibles less liquid than conventional bonds.

Contingent Convertible bonds (CoCos), also referred to as enhanced capital notes (ECNs) constitute another variant. While both instruments involve the conversion of debt into equity, they differ fundamentally in terms of trigger mechanisms, investor control, and intended function within the financial system.
CoCos are designed primarily as regulatory capital instruments. They automatically convert into equity when a specific trigger is met, usually when the issuing bank's capital ratio falls below a defined threshold. Unlike standard convertibles, CoCos do not offer conversion at the investor's discretion. Instead, conversion is imposed under adverse conditions, often resulting in the receipt of equity at depressed valuations. The main function of CoCos is to enable financial institutions, particularly banks, to absorb losses and maintain solvency without external support. These instruments gained prominence following the 2007–2008 financial crisis, when they were introduced as a mechanism to strengthen bank capital structures and reduce the need for taxpayer-funded bailouts (De Spiegeleer and Schoutens, 2014).
From an investment perspective, standard convertibles offer a more favourable risk-return profile under normal market conditions, combining steady cash flows with upside equity participation. CoCos, on the other hand, carry a higher level of risk, as their conversion is typically triggered in times of financial distress. If the trigger condition is never met, CoCos may be redeemed at maturity in the same way as conventional bonds. The convertible bond market represents a significant and growing segment in the landscape of fixed-income financial instruments. In recent years, the global convertible bond market has seen a substantial expansion. According to Bloomberg® data and other market sources, the total amount of convertible bond issuances reached $150 billion in 2021, an increase from previous years, driven by favourable market conditions and growing demand for hybrid instruments. S&P Global reports that 2021 was a particularly dynamic year for new issuance, fuelled by market volatility that incentivised companies to seek flexible financing arrangements. Among the most active sectors are technology and healthcare, where companies often need capital to support innovation and long-term growth. On the investor side, convertible bonds are valued for their ability to mitigate risk. With the bond component, investors enjoy downside protection while maintaining a fixed return, while the conversion component offers upside potential if the share price rises. Morningstar points out that funds specialising in convertible bonds have been able to outperform traditional fixed-income funds in good market times, thanks to the ability to participate in stock market rises.
Geographically, the convertible bond market is dominated by the US, Europe and Asia, with the US accounting for the largest share of issues (Calamos Investments, 2021). In Europe, issuances are mainly driven by the banking and technology sectors, where CoCo (Contingent Convertible Bonds) instruments play a key role in banks' regulatory capital. In Asia, countries such as Japan and Hong Kong have developed well-regulated and growing convertible bond markets, with strong interest from both local issuers and global investors.

2) Fair value determination of a Convertible Bond

Convertible bonds embed characteristics of both debt and equity, making their valuation challenging. The presence of multiple embedded option and early-exercise features often makes closed-form solutions unsuitable, especially within realistic market conditions.
In early works by Ingersoll (1977), such solutions rely on assumptions of market completeness and continuous-time trading, which severely limit their applicability. Further critiques by Nyborg (1996) highlighted the inability of analytical models to incorporate market frictions, non-linear payoff structures, and conditions such as early calls or forced conversions.
This complexity led to the development of numerical approaches, particularly within two dominant theoretical frameworks: the structural and the reduced-form approaches.
The structural approach, developed by Ingersoll (1977) and Brennan and Schwartz (1980), models the bond as a contingent claim on the firm's value, assuming the default occurs when the firm's asset value falls below a given value. This method assumes a simplified capital structure, continuous monitoring and perfect observability of firm value, greatly limiting its practical relevance. Furthermore, estimating volatility on a firm-level asset base is commonly unfeasible, especially when other senior claims coexist (Brennan and Schwartz, 1980).
The reduced-form approach, theorized by McConnell and Schwartz (1986), assumes the convertible bond as a contingent claim on the underlying stock price, where credit risk is modeled exogenously. Early implementations, such as Ho and Pfeffer (1996), used a risk-adjusted constant credit spread to reflect default risk, but this assumption has been criticized for failing to capture time-varying credit conditions.
A key contribution in this space was the Tsiveriotis and Fernandes (1998) model, which decomposes a convertible bond into a fixed-income part (subject to credit risk) and an equity-linked part (considered default-free), each discounted with different rates. This model became widely adopted due to its tractability and alignment with market practice.
Refinements by Ayache, Forsyth and Vetzal (2003) and Gushchin and Curien (2008) introduced endogenous modeling of default probability and recovery rates. The former in particular embedded credit risk within both the equity and debt components and incorporated partial recovery, producing more accurate pricing compared to the Tsiveriotis and Fernandes

(1998) and Brennan and Schwartz (1980) models in empirical comparisons. Gushchin and Curien (2008) also demonstrated that modeling credit spreads as stochastic processes, rather than constant, improved calibration and reduced errors.

Given the multidimensional risk profile of convertible bonds, numerical methods are nowadays the dominant paradigm.

Among these, lattice-based methods (or stochastic tree models) play a central role due to their intuitive structure and ability to capture discrete events and decision points such as coupon payments, callability, or conversion. These methods discretize time and the evolution of relevant state variables across a grid, where the bond value is recursively computed from maturity to the present, reflecting the expected payoff under each possible path.

Hung and Wang (2002) developed a binomial tree model incorporating stochastic interest rates via the Ho and Lee (1986) model and default risk through the Jarrow and Turnbull (1995) framework, assuming the stock price jumps to zero upon default. They assigned probabilities to stock price, interest rate, and default transitions, concluding that such integration improved pricing accuracy. Chambers and Lu (2007) refined this approach by introducing a non-zero correlation between stock returns and interest rates, an often-overlooked dependency, using the same interest rate and credit models. Their results underscored the importance of jointly modeling correlated risks in convertible bond valuation.

Das and Sundaram (2007) incorporated stochastic volatility using the Constant Elasticity of Variance (CEV) model, along with stochastic interest rates (modeled à la Heath, Jarrow, Morton, 1990) and default risk. Their framework allowed for correlation between equity and term structure dynamics and showed that the sensitivity of bond value to credit risk is magnified under high volatility conditions.

Ho and Pfeffer (1996) used a binomial model with deterministic volatility and interest rates but introduced credit risk through an option-adjusted spread. Their study on US callable convertible bonds highlighted systematic underpricing in the market.

Trinomial trees have also been adopted to improve convergence and computational accuracy. Gushchin and Curien (2008) used a trinomial tree within the Tsiveriotis-Fernandes framework and found their approach capable of handling a wider range of instruments and market conditions.

Rotaru (2006) applied another trinomial method and modeled callable convertibles using deterministic volatilities and credit spreads and observed consistent underpricing across different market segments.

Overall, tree models have proven to be among the most effective methods in handling the discrete features and early-exercise optionalities typical of convertible bonds. Their flexibility allows for the modeling of complex structures. While simulation methods, such as Monte Carlo approaches, are often preferred for higher-dimensional problems and deeply path-dependent features, tree-based methods remain a robust and intuitive alternative, that balances accuracy and computational feasibility.

Despite these studies, significant gaps remain.

Most tree models rely on simplifying assumptions such as deterministic volatility, constant credit spreads, or uncorrelated risk factors, which reduce realism. Relatively few contributions have succeeded in jointly integrating stochastic equity, interest rates, and credit spreads within a single computationally tractable lattice framework. The consistent empirical evidence of underpricing across various markets suggests that the valuation of convertible bonds remains an open research area. These observations point toward the need for hybrid or adaptive numerical schemes that can reflect complex market imperfections while maintaining robustness and interpretability.

The pricing of convertible bonds has traditionally relied on established numerical methods such as finite difference models, finite element methods, binomial or trinomial trees and Monte Carlo methods. However, the adoption of alternative tree models remains relatively limited, indicating a potential gap in the research. Recent studies started to explore these alternative models.

Hu, Li and Liu (2022) introduced a Jarrow-Rudd model that incorporates asymmetry and kurtosis in the returns of the underlying asset, using an asymmetric random walk process.

This approach allows for more accurate calibration to implied volatility surfaces and includes hedging costs, bringing the model closer to real market scenarios. The foundation for this line of modeling can be traced back to Jarrow and Rudd (1986), who proposed a simplified binomial tree based on equal probabilities for upward and downward price movements. Although computationally efficient, their approach lacked the risk-neutral valuation framework, which limited its practical application in derivative pricing, Jarrow and Rudd (1986). Despite its applicability, the use of the JR model is not yet widespread in this context. Similarly, Tian proposed a flexible binomial tree that matches not only the mean and variance but also the skewness of the underlying asset return distribution. This enhancement improves convergence and accuracy in option pricing, particularly for instruments sensitive to higher-order moments (Tian, 1993a and Tian, 1993b). Haahtela (2006) extended this line of development by introducing a trinomial tree framework designed to incorporate informed trading and asymmetric information, offering a structure more aligned with realistic market behaviour.

Milanov and Kounchev (2012) developed a binomial tree model for pricing convertible bonds that accounts for credit risk, demonstrating convergence to the model of Ayache, Forsyth and Vetzal (2003). This model integrates both reduced and synthetic approaches for modeling default risk, offering an alternative to traditional models.

Despite these developments, the adoption of alternative tree models in convertible bond pricing remains marginal. Most studies continue to focus on traditional lattice models, which have limitations in incorporating more complex market features. There is a significant opportunity to further explore and develop alternative tree models in the context of convertible bonds. These models could offer greater flexibility and accuracy in valuation, especially in complex market scenarios or when dealing with non-standard optional features.

In the next section, we discuss the most widely used valuation methodology for convertible bonds, namely the Cox, Ross and Rubinstein stochastic binomial tree (CRR Tree). The theoretical foundation for this approach was initially proposed by Cox and Ross (1976), who explored the valuation of options under alternative stochastic processes. In the fourth section, we then review the three most popular alternative binomial trees: Haahtela, Jarrow-Rudd and Tian. We first explain the theoretical principles and peculiarities of each approach, then implement them in pricing European and American options.

European options allow the holder to exercise the option only at expiration, while American options offer greater flexibility by allowing exercise at any time up to expiration. This flexibility makes American options more complex to price, as it requires taking advantage of early exercise opportunities.This step is considered to be preparatory to the application of these techniques to the quantitative analysis of convertible bonds because it provides evidence of the correct implementation of the method. Once we verify that the alternative stochastic binomial trees converge correctly, we can implement them for the valuation of a convertible bond. In the fifth section, we then show how to adapt the traditional CRR numerical scheme to alternative techniques: after illustrating the step-by-step procedure and highlighting implementation differences, we provide proof of correct implementation through a convergence analysis. In order to conduct a complete quantitative analysis, we also numerically estimate the sensitivity of the models to the main risk parameters, i.e., the change in the underlying (Delta and Gamma) and volatility (Vega). In this analysis, the Jarrow-Rudd tree has shown a different sensitivity than the other approaches on Vega.

In the sixth section, we provide empirical evidence for our findings by offering numerous market cases suitable for confirming the analysis performed. Robustness in the estimation of the main quantitative measures that can be associated with a convertible bond for all the different alternative stochastic binomial trees discussed in the study is thus proven.

3) The most widespread pricing methodology for pricing a Convertible Bond: the CRR Tree

We provide a short explanation of the working principle of the Cox, Ross and Rubinstein Tree in Appendix A, together with a proof of the derivation of the three main parameters (the up factor: u, the down factor d and the probability $\prod$) that rule the projections of the underlying. We assume the up and down parameters to be constant throughout the paper.

The mathematical notation used for the description is the standard notation adopted by (Haug, 2007) and briefly described in the same Appendix. Consequently, this paragraph focuses exclusively on the implementation of this technique in the convertible bond pricing.

As we have already explained, a convertible bond can be viewed as a combination of a traditional bond and a stock option. When the stock price is significantly lower than the conversion price, the convertible bond behaves like a simple bond. Conversely, when the share price is much higher than the conversion price, it behaves more like a stock. This dynamics should influence the discounting of cash flows.

In a risk-neutral setting, this does not mean arbitrarily changing the discount rate; rather, it reflects the fact that the appropriate discount rate depends on the nature of the payoff being replicated. For a deeply out-of-the-money convertible, future cash flows should be discounted at a rate that includes the credit spread k above the bond's Treasury rate.

If the convertible is deeply in-the-money, the conversion is almost certain, and the cash flows should be discounted at the risk-free rate. Bardhan et al. (1994) incorporated these considerations by applying a discount rate based on a variable conversion probability.

The CRR Tree model begins with a standard binomial stock price tree.

The convertible bond price is then calculated by working back from the final nodes of the stock price tree, ensuring that the value of the convertible at each end node equals the greater of the conversion value or the face value plus the final coupon. To roll backward through the tree, backward induction is used.

If it is optimal to convert the bond, the value is set equal to the conversion value at that node, or else the convertible bond value $P_{n,i}$ is set equal to:

$$P_{n,i} = \max\left[mS, \prod P_{n+1,i+1} e^{-r_{n+1,i+1}\Delta t} + (1-\prod) P_{n+1,i} e^{-r_{n+1,i}\Delta t}\right] \quad (1)$$

where *m* represents the conversion ratio. Certain convertible bonds include an initial lockout period during which conversion is not permitted. At these nodes, the convertible bond value can be simplified to:

$$P_{n,i} = \prod P_{n+1,i+1} e^{-r_{n+1,i+1}\Delta t} + (1-\prod) P_{n+1,i} e^{-r_{n+1,i}\Delta t} \quad (2)$$

Rather than applying a constant discount rate r, the discount rate $r_{n,i}$ is adjusted to vary with the conversion probability $q_{n,i}$ at each node. The conversion probabilities $q_{n,i}$, where *n* is the time step and *i* is the number of up moves (state), are determined by working backward from the end of the stock price tree.
If conversion is optimal at a given end node, the conversion probability is set to 1; otherwise, it is set to 0. For earlier time steps, the conversion probability is also set to 1 if it is optimal to convert at that node; otherwise, it remains at:

$$q_{n,i} = \prod q_{n+1,i+1} + (1 - \prod) q_{n+1,i} \quad (3)$$

The credit-adjusted discount rate is set equal to a conversion probability weighted mixture of the risk-free rate and the credit-adjusted rate. This gives a discount rate for up moves equal to:

$$r_{n,i} = q_{n,i} r + \left(1 - q_{n,i}\right)(r + k) \quad (4)$$

The discount rate is therefore set to the constant risk-free rate r when the conversion probability is 1, and to $r + k$ (the risk-free rate plus the credit spread) when the conversion probability is 0. For conversion probabilities between 0 and 1, the discount rate transitions smoothly between the risk-free and credit-adjusted rates.

Example

In this subsection, we analyse a traditional pricing of a Convertible bond, Haug (2007): we generated the following trees using the data provided in the book and implemented it in a more efficient numerical environment (Python).
Let us consider a convertible corporate bond with five years to maturity. The continuously compounding yield on a five-year treasury bond is 7%, the credit spread on the corporate bond is 3% above treasury, the face value is 100, the annual coupon is 6, the conversion ratio is 1, the current stock price is 75, and the volatility of the stock is 20%.
Consequently, the inputs of the model are: $S = 75$, $T = 5$, $r = b = 0.07$, $k = 0.03$, $m = 1$, and $\sigma = 0.20$.
To price the convertible bond, we need to build a standard binomial stock price tree. With the number of time steps $n = 5$, we obtain Δt = 1 and up and down factors are:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{1}} = 1.2214 \qquad d = \frac{1}{u} = 0.8187$$

The probability of an increase in price is thus given by:

$$\prod = \frac{e^{b\Delta t} - d}{u - d} = \frac{e^{0.07\times 1} - 0.8187}{1.2214 - 0.8187} = 0.6302$$

and we obtain the binomial stock price tree in Figure 1.

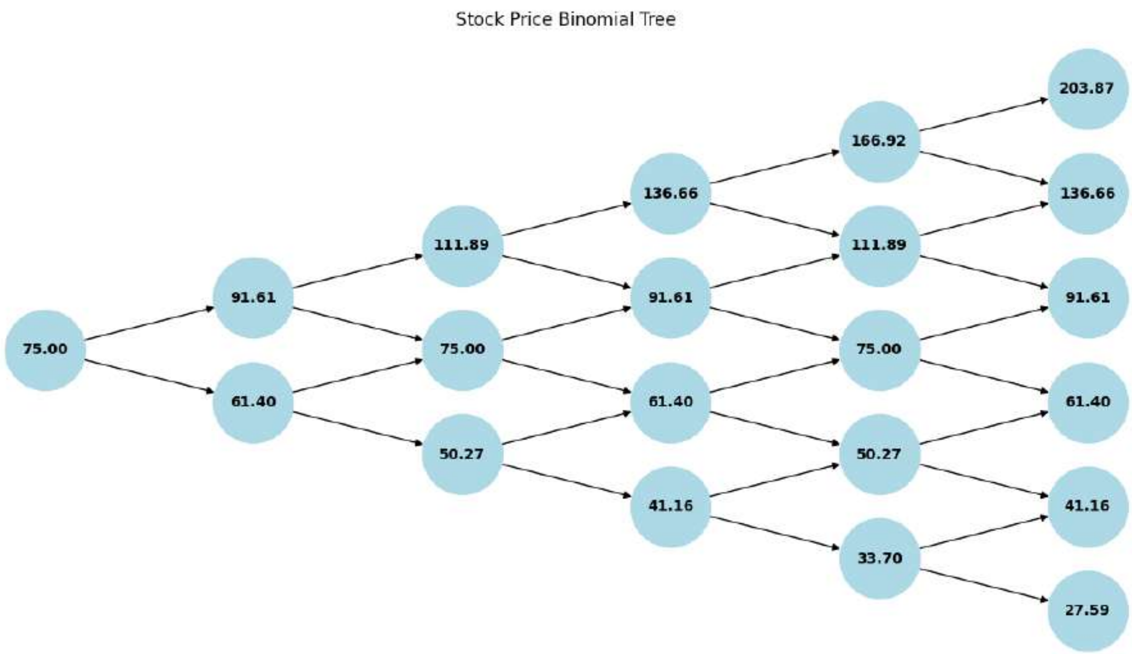

Figure 1: 5-steps Stock Price Binomial Tree: numeric example

The next step is to find the convertible bond values and the conversion probabilities at each node in the tree. Firstly, let us look at the calculation of several nodes.
At the end node with stock price 203.87, it is better to convert the bond into one stock and receive the stock price 203.87 rather than get the notional plus the coupon (100 + 6). The probability of conversion at this node, $q_{5,5}$, is 100%, which we write as 1.00 in the conversion probability tree.
At the end node, with a stock price of 91.61, it is better not to convert the bond and receive the face value plus the coupon of 106. The probability of conversion is $q_{5,3} = 0$. For the node at year four (n = 4) with stock price 111.89, the convertible bond value of 121.77 is found using equation (1):

$$P_{4,4} = \max[1 \times 111.89, 0.6302 \times 136.66 e^{-r_{n+1,i+1} \times 1} + (1 - 0.6302) 106.00 e^{-r_{n+1,i+1} \times 1}]$$

The credit-adjusted discount rates are found using equation (4):

$$r_{n+1,i+1} = 1 \times 0.07 + (1 - 1)(0.07 + 0.03) = 0.07$$

$$r_{n+1,i} = 0 \times 0.07 + (1 - 0)(0.07 + 0.03) = 0.1$$

The conversion probability of 0.63 at this node is given by equation (3):

$$q_{4,4} = 0.6302 \times 1 + (1 - 0.6302) \times 0 = 0.6302$$

The same procedure can be used to find any convertible bond value and conversion probability.

The previous section outlined the basic principles of how to incorporate a convertible bond model. In practice, there are many other aspects to consider. Some convertible bonds allow the issuer to force investors to convert the bond if the share price reaches a certain predetermined level (barrier).
To include a barrier in the convertible binomial model, the number of time steps must be chosen so that the barrier falls exactly on the nodes.
The conversion probability is then set to 1 if the share price is greater than or equal to the barrier. The issuer of the convertible bond often has the right to call the bond, while the investor has the right to sell the bond. Figures 2 and 3 in this paragraph illustrate the 5-steps stock price and convertible bond value trees, respectively.

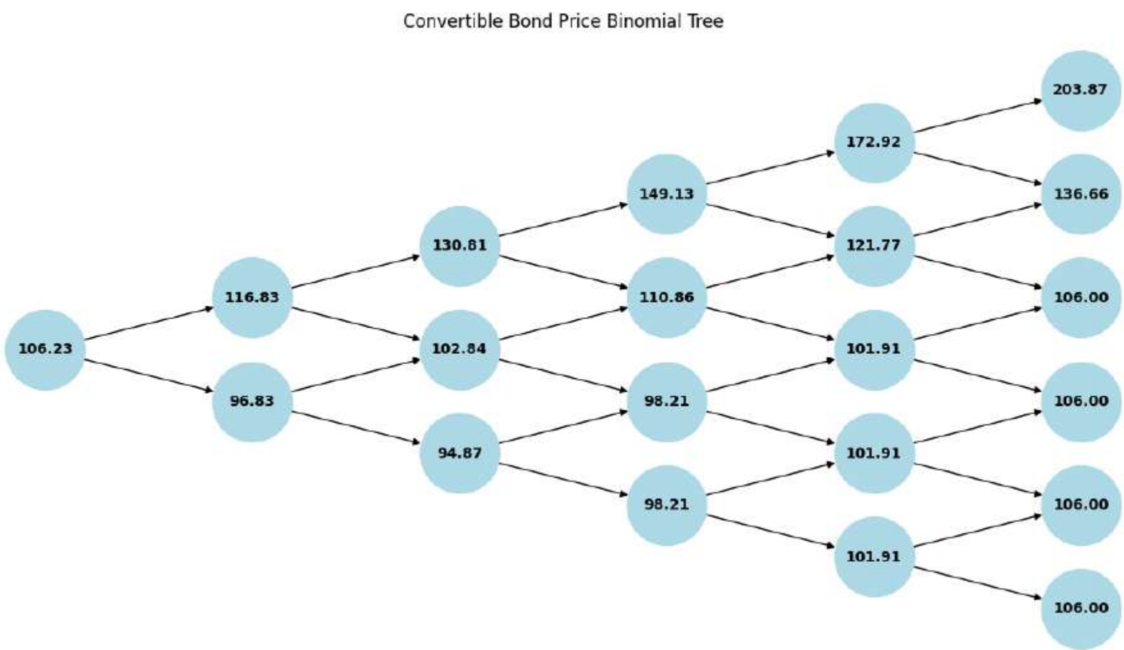

Figure 2: 5-Steps Convertible Bond Price binomial Tree: numeric example

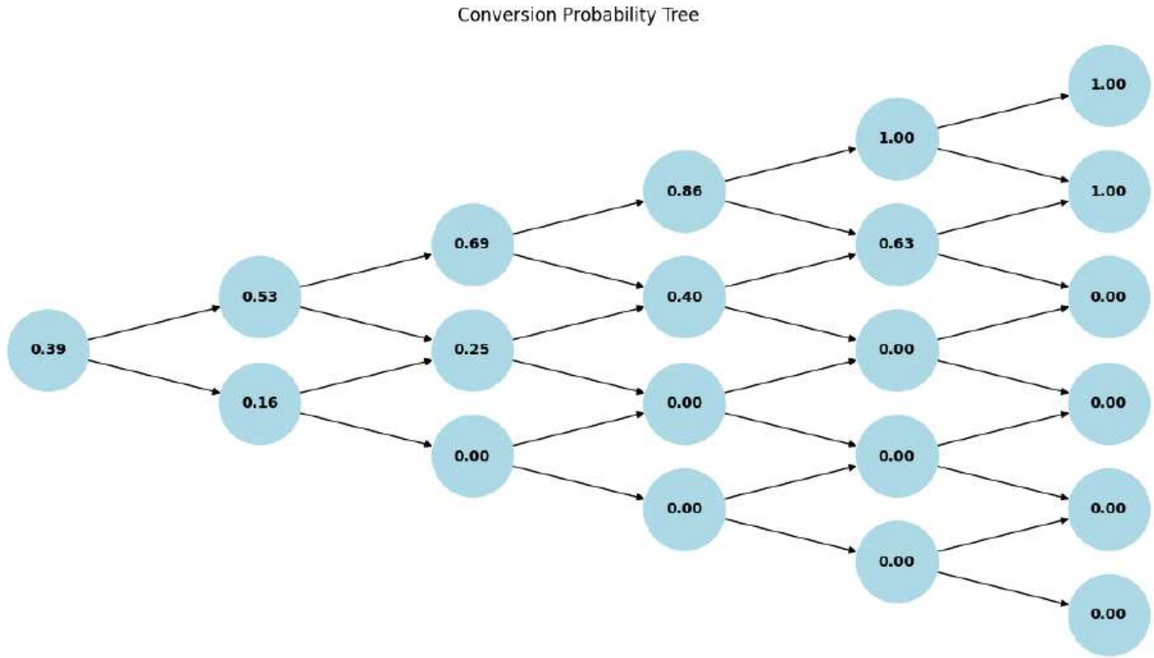

Figure 3: 5-Steps Conversion Probability Tree: numeric example

Convergence of Binomial Trees for CRR model

Having introduced the valuation of European and American options using the binomial tree method, we now examine the accuracy of this technique.

The convergence analysis of the binomial tree is fundamental to understand its effectiveness in option pricing. Our analysis examines how the binomial tree approximates the actual option price as the number of time steps increases.

When analysing convergence, we need to consider the error of a numerical scheme.

If V_{exact} represents the correct option value and V_n is the value from a binomial tree with n steps, the error can be expressed as:

$$Error_n = V_{exact} - V_n \quad (5)$$

To formally define convergence, there exists a constant k such that, for all time steps n:

$$Error_n = O\left(\frac{1}{n^c}\right) \quad (6)$$

where c is the order of convergence. As long as $c > 0$, V_n will converge to V_{exact}. Mathematical proof that shows the convergence of the binomial lattice to the true option price is described in (Giribone and Ventura, 2011).

For European options, we can empirically examine this convergence since we have an analytic expression for V_{exact} (the Black-Scholes price).

We know that the binomial distribution will eventually converge to the lognormal distribution, which underlies the Black-Scholes model.

Empirical evidence shows that, for all basic binomial models (e.g., CRR, RB) $c = 1$, meaning V_n converges to the Black-Scholes price at a rate of $\frac{1}{n}$.

In general, to halve the error, the number of time steps needs to be doubled (Leisen and Reimer, 1996).

In our case, as shown in Figure 4, the convergence chart represents the price behaviour of a convertible bond using a binomial tree approach with the Cox-Ross-Rubinstein model.

The x-axis denotes the number of steps (N) and the y-axis represents the computed price of the convertible bond.

We can see there is a rapid initial convergence from step 3 to around 15 steps. At a low number of steps, the bond price shows significant changes, rising sharply to around 130.

After reaching around 20 steps, the bond price begins to stabilise near 130, indicating convergence.

Beyond this point, minor oscillations persist, fluctuating slightly above and below the convergence value. The plot in Figure 4 suggests that the model reaches practical convergence after about 20 steps.

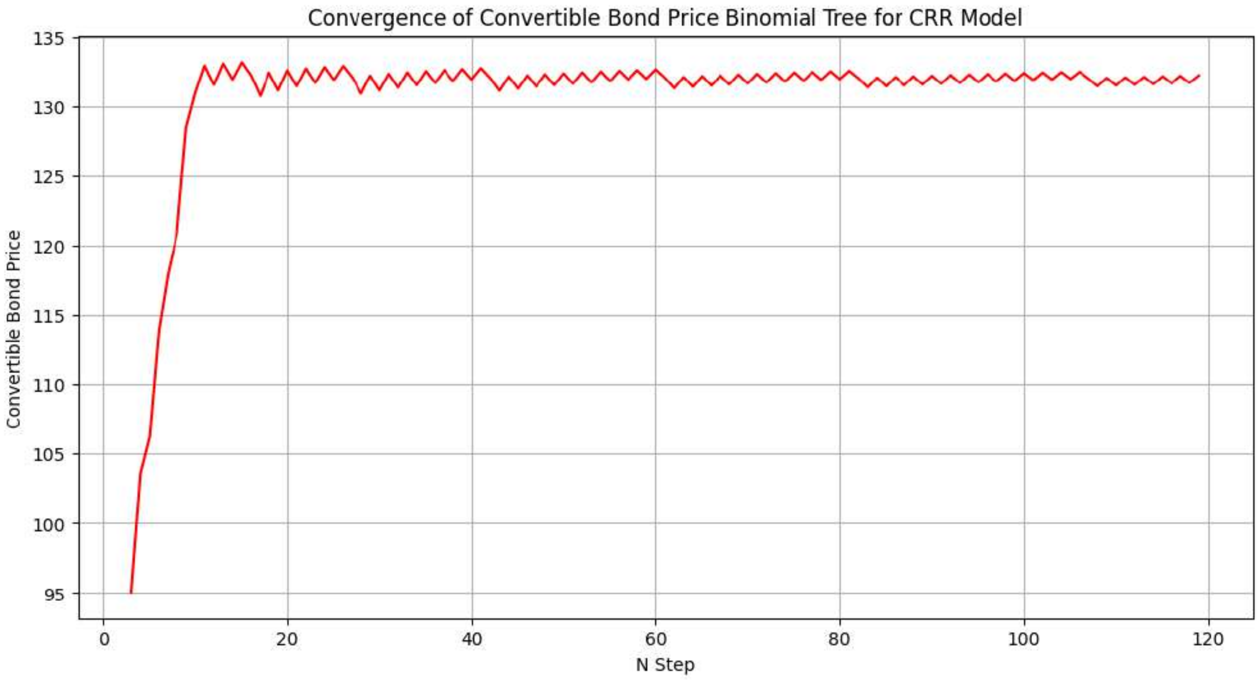

Figure 4: Convergence of convertible bond price for the CRR binomial tree model

4) Alternative binomial trees

We now conduct a convergence analysis of the CRR, Tian, Jarrow-Rudd, and Haahtela models. Our goal is to evaluate the stability and accuracy of each model when pricing convertible bonds by systematically increasing the number of time intervals. As described in the Introduction, all models are calibrated to ensure consistency with the statistical properties of the underlying asset, such as first and second moments.

Tian Binomial Tree Model

The Tian model improves the classic binomial lattice for option pricing by modifying parameters to achieve greater accuracy in approximating the continuous-time stochastic process of an underlying asset. Specifically, the Tian approach adjusts the up and down factors in the lattice so that the lattice corresponds to the first three moments of the continuous-time distribution: mean, variance and skewness. This additional moment constraint is a key distinction from traditional models such as the Cox-Ross-Rubinstein model, which typically only corresponds to the first two moments (mean and variance).
The stock price in a binomial lattice can either increase by a factor u or decrease by a factor d in each time period Δt, where $\Delta t = \frac{T}{N}$ denotes the discrete time step (also referred to as a bump). T is the time to maturity, with N discrete steps. Tian defines the up and down factors using the growth factor $M = e^{r\Delta t}$ which corresponds to the drift under a risk-neutral measure, and the volatility adjustment $v = e^{\sigma^2 \Delta t}$. Then, the model proposed by Tian matches the first three moments of the log-normal distribution followed by the underlying:

$$u = \frac{1}{2} e^{b\Delta t} v \left(v + 1 + \sqrt{v^2 + 2v - 3} \right) \quad (7)$$

$$d = \frac{1}{2} e^{b\Delta t} v \left(v + 1 - \sqrt{v^2 + 2v - 3} \right), \qquad v = e^{\sigma^2 \Delta t} \quad (8)$$

$$\prod = \frac{e^{b\Delta t} - d}{u - d} \quad (9)$$

With these values, the binomial model distribution converges to the lognormal distribution of continuous-time stock prices more accurately by accounting for skewness (third moment). This skewness correction helps the Tian model better approximate the behaviour of asset prices, especially in fewer steps compared to traditional binomial models.
Below, we show the values obtained relative to the auxiliary variables u, d and $\prod$ for the Tian model with respect to the example shown in section 3.

$$Model: Tian \longrightarrow u = 1.3657, \qquad d = 0.9124, \qquad \prod = 0.3532$$

The stock price tree generated by the Tian model reflects a wider, more skewed distribution of stock prices over time, accounting for realistic price volatility and potential skewness (see Figure 5). Each node shows a potential stock price at each time interval, with branches indicating upward or downward movements. By incorporating skewness into the Tian model, we observe a wider range of stock prices compared to simpler models, particularly at the terminal nodes, which capture more extreme highs and lows.

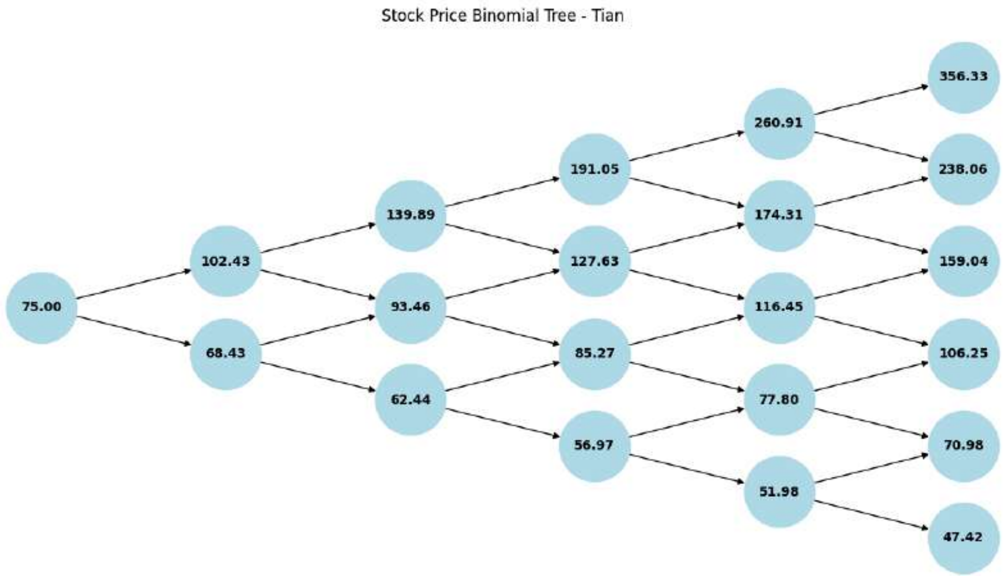

Figure 5: 5-Steps Stock Price Binomial Tree – Tian Model

Jarrow-Rudd Tree Model

The Jarrow-Rudd model is an extension of the traditional binomial approach, and it introduces modifications to the tree model structure that can more accurately adapt to market conditions. Like other binomial option pricing models, Jarrow-Rudd binomial trees are characterized by the sizes of up and down moves and their associated probabilities. The main feature of the Jarrow-Rudd model is that up and down moves are chosen to have a probability of $\frac{1}{2}$.

Beyond these specific formulas, the rest of the model aligns with the Cox-Ross-Rubinstein and other binomial option pricing models.

The up and down multipliers u and d are used to construct the underlying price tree, starting at the current underlying price S and extending to the option expiration. At each node in the tree, the price can either move up $S \cdot u$ or down $S \cdot d$ at either node in the next step.

The underlying prices at expiration (the last nodes in the tree) are then used to determine the option payoffs, forming the last layer of the option price tree. From there, the option price tree is calculated backwards, working toward the first node, which gives the current option price.

$$u = e^{\left(b - \frac{\sigma^2}{2}\Delta t + \sigma\sqrt{\Delta t}\right)} \quad (10)$$

$$d = e^{\left(b - \frac{\sigma^2}{2}\Delta t - \sigma\sqrt{\Delta t}\right)} \quad (11)$$

$$\Pi = \frac{1}{2} \quad (12)$$

Where $b = r - q$ is the so-called “cost-of-carry” (see Appendix A).

The equal probability assumption simplifies calculations and aligns with certain market conditions where no bias exists in price movements.

Below, we show the values obtained relative to the auxiliary variables u, d and p for the Jarrow-Rudd model:

$$Model: Jarrow - Rudd \longrightarrow u = 1.2840, \quad d = 0.8607, \quad \Pi = 0.5000$$

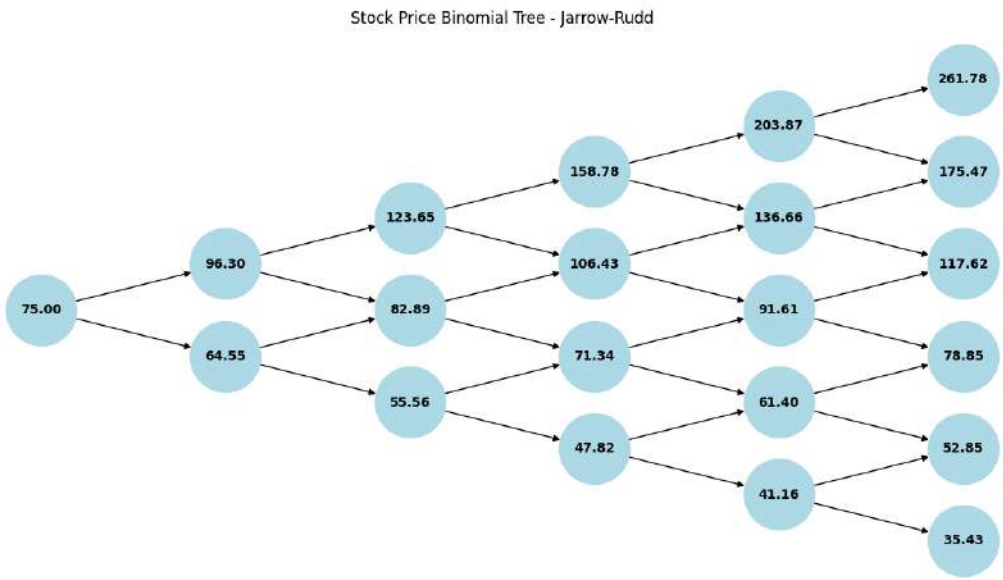

Figure 6: 5-Steps Stock Price Binomial Tree – Jarrow Rudd

In Figure 6 representing the stock price tree, we observe a symmetrical progression of prices as we proceed through the time steps. This symmetry stems from the structure of the Jarrow-Rudd model, which adjusts the risk-neutral drift by maintaining a normal-type distribution around the expected stock price.

Haahtela Tree Model

The Haahtela model addresses option pricing where the underlying asset distribution diverges from the traditional lognormal shape by introducing flexibility for distributions that can incorporate both positive and negative values. This model is particularly useful when the underlying asset value exhibits dynamics that are not well modeled by Geometric Brownian Motion.
In traditional Geometric Brownian Motion based binomial trees, the up and down factors for each step are calculated using the volatility parameter to approximate the future asset price distribution as lognormal.
However, the Haahtela model introduces a shifted diffusion process different than the one of the Geometric Brownian Motion binomial trees, which allows the underlying value to follow a path between normal and lognormal distributions.
This is achieved by introducing a "shift" or "displacement" parameter, which allows the model to capture the skewness and better handle negative values, when necessary.
The Haahtela model adapts the up and down factors and the probability $\prod$ as follows:

$$u = e^{b\Delta t}\left(1 + \sqrt{e^{\sigma^2 \Delta t} - 1}\right) \quad (13)$$

$$d = e^{b\Delta t}\left(1 - \sqrt{e^{\sigma^2 \Delta t} - 1}\right) \quad (14)$$

$$\prod = \frac{e^{(r-q)\Delta t} - d}{u - d} \quad (15)$$

These values are adjusted to account for the shifted distribution by setting the increase factor u and the decrease factor d based on both the traditional volatility and the shift parameters.
The probability of an upward move $\prod$, is modified to align with the risk-neutral valuation principle while accommodating the shifted process.
This formulation maintains no-arbitrage conditions within the model while reflecting the altered dynamics of the displaced process. Below, we show the values obtained relative to the auxiliary variables u, d and $\prod$ for the Haahtela model:

$$Model: Haahtela \longrightarrow u = 1.2892, \qquad d = 0.8558, \qquad \prod = 0.5000$$

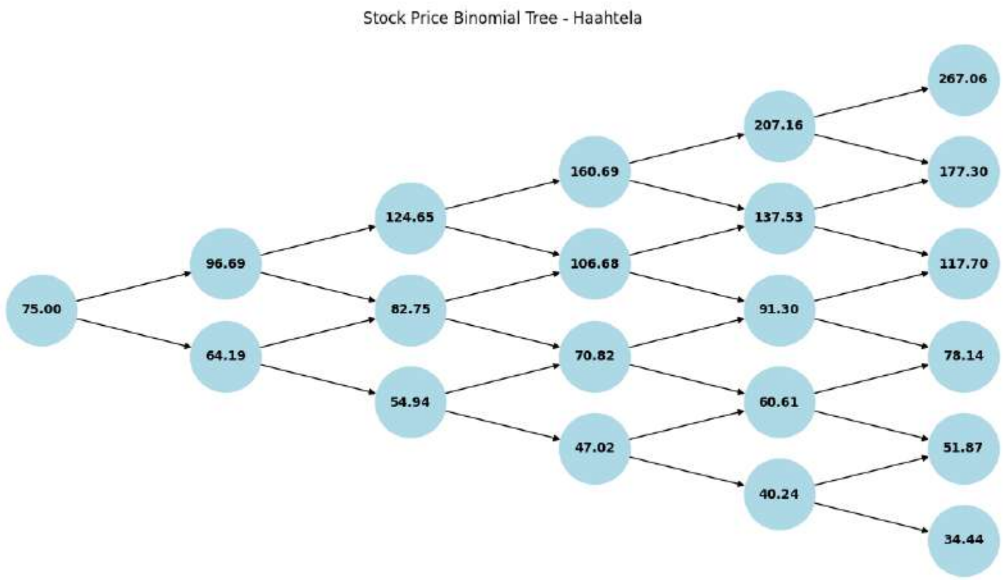

Figure 7: 5-Steps Stock Price Binomial Tree – Haahtela Model

Figure 7 illustrates the evolution of share prices using the specific parameters of the Haahtela model, which tends to allow a wider range of potential outcomes for each node. This tree structure suggests a higher level of implied volatility, as shown by the wide gap between the highest and the lowest final price.

Indeed, the initial stock price starts at 75.00. Moving up the tree, there are rapid increases, reaching a high of 267.06 at the highest final node, indicating strong upside potential according to the assumptions of the Haahtela model. In contrast, the lowest end node ends at 34.44, showing significant downside potential.

This large gap between the highest and the lowest possible price indicates that the parameters of the Haahtela model could incorporate a higher volatility factor, allowing for more pronounced stock price fluctuations over time.

Such a model might be suitable for activities characterised by greater uncertainty or speculative behaviour, where price fluctuations are expected to be more extreme.

The structure reflects the model flexibility in capturing upside and downside risks, which could be useful in the valuation of volatility-sensitive options or derivatives.

Convergence Analysis of the Alternative Binomial Trees

This section aims to compare an alternative pricing model with the CRR model in order to validate its accuracy and reliability for the pricing of both European and American call and put options.

The CRR model has been extensively tested and refined for both types of options, making it a suitable benchmark for assessing the convergence and accuracy of other pricing models.

In this paragraph we analyse the performance of our model by comparing the prices of European and American call and put options with those generated by the CRR model.

Using the same parameters for both models, we examine the degree of convergence between the models, as the number of steps in the binomial tree increases.

We perform the computation using the following parameters: $S_0 = 100, K = 95, T = 0.5$ years, $r = b = 0.01$, and $\sigma = 0.20$. We use a European call option because it has a simple structure and can only be exercised at maturity, making it easier to compare with the theoretical Black-Scholes price.

For the American option, we chose a put since its value is more affected by early exercise, especially when the stock price decreases.

This combination of a European call and an American put allows for a comprehensive convergence analysis. The European call serves to validate the numerical model's accuracy and reliability under simpler conditions, while the American put tests the model's ability to handle the additional complexity of early exercise.

This benchmarking and convergence analysis will provide insights into the robustness and potential limitations of our model with respect to an established standard.

First, we calculate the theoretical price of a European call and put option using the Black-Scholes. The BS framework provides a closed form solution based on inputs like the stock price, strike price, time to maturity, risk-free rate, and volatility. These results serve as a benchmark for assessing the accuracy of the numerical methods.

Then we move on to numerical techniques, specifically binomial tree models, to approximate the option prices.

To study the performance of these models, we conducted a convergence analysis. We applied the numerical methods to both European and American options.

For European options, we calculated the value through backward induction, starting from the terminal payoff at maturity and moving step by step to the present.

For American options, the code incorporates the possibility of early exercise so we checked, at each step, whether exercising the option was more favourable than holding it. This involves comparing the intrinsic value of the option with its continuation value.

We recorded and plotted the results from each model, showing that the estimated prices converge toward the theoretical Black-Scholes price. This provides valuable insights into the speed and accuracy of each model (see Figure 8).

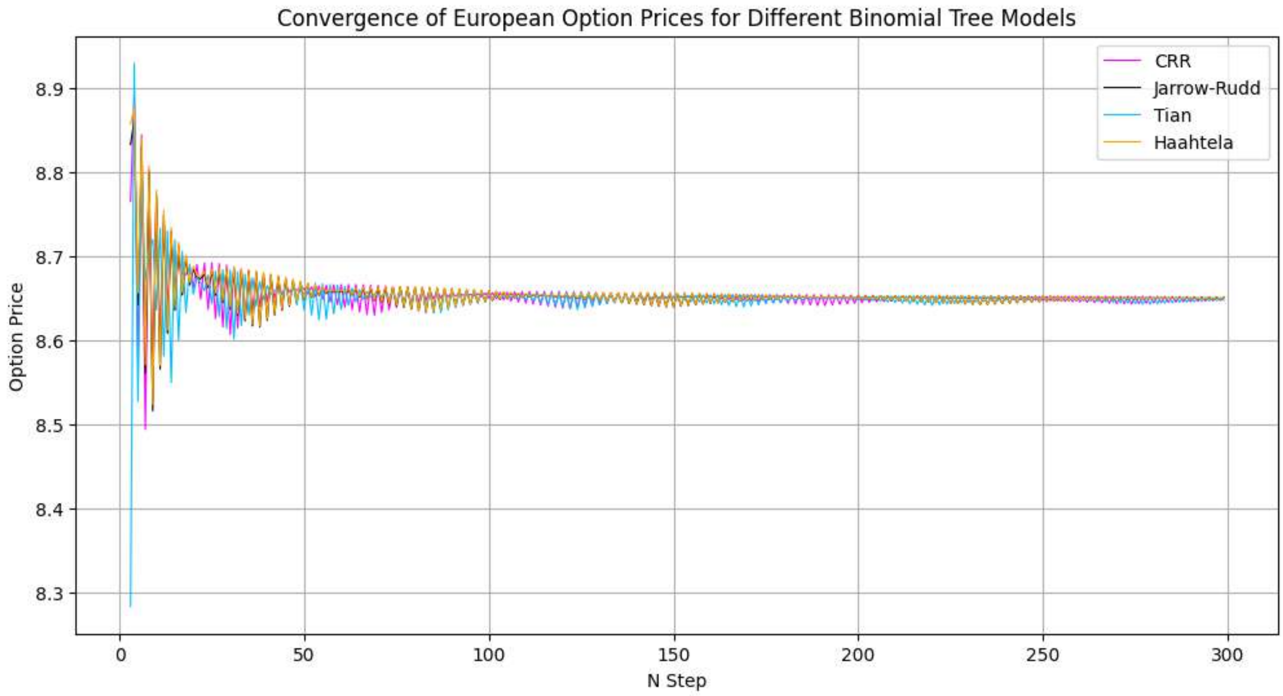

Figure 8: Convergence of European Option Prices for Different Binomial Tree Models

The plot illustrates the convergence of European option prices calculated with different binomial tree models as the number of steps (N Steps) in the tree increases.

Each plot represents a different model: CRR, Jarrow-Rudd, Tian and Haahtela. The y-axis shows the option price, while the x-axis shows the number of steps in the binomial tree.

Initially, there is a significant fluctuation in option prices with a low number of steps, which is particularly noticeable in models such as Tian.

As the number of steps increases, these fluctuations reduce, and the models begin to converge towards a stable price. Around 150-200 steps, all models begin to align closely, indicating that they are approaching the theoretical value predicted by the Black-Scholes formula.

This pattern of convergence suggests that a higher number of steps leads to greater accuracy in binomial tree models. However, the speed of convergence and initial stability vary depending on the model.

For example, the Tian model exhibits greater initial fluctuations, while the CRR shows relatively more uniform convergence.

The plot shows that all models eventually stabilise around the same price, validating the numerical methods against the analytical benchmark, as the tree size increases.

Additionally, we analysed the impact of the number of steps on the precision of American options, which are more complex due to the early exercise feature.

The visualization offers a clear comparison between the models. For instance, the CRR model is expected to provide consistent results as the number of steps increases, but alternative methods like Tian or Haahtela may show faster convergence or better accuracy under certain conditions. These observations help highlight the trade-offs between computational complexity and precision.

Figure 9 shows the convergence of American option prices calculated using different binomial tree models, as the number of steps (N Step) in the tree increases.

American option prices are slightly higher than the corresponding European prices due to the possibility of early exercise inherent in American options.
The possibility of early exercise introduces additional complexity, making the convergence process slightly more variable compared to European options.
Overall, the plot shows that although the models behave differently at a lower number of steps, they all converge to a similar value as the number of steps increases.
This validates the robustness of these binomial models in pricing American options, while highlighting the patterns of convergence and the impact of early exercise characteristics.
Consistent results at higher numbers of steps reinforce the reliability of numerical methods for accurate pricing.

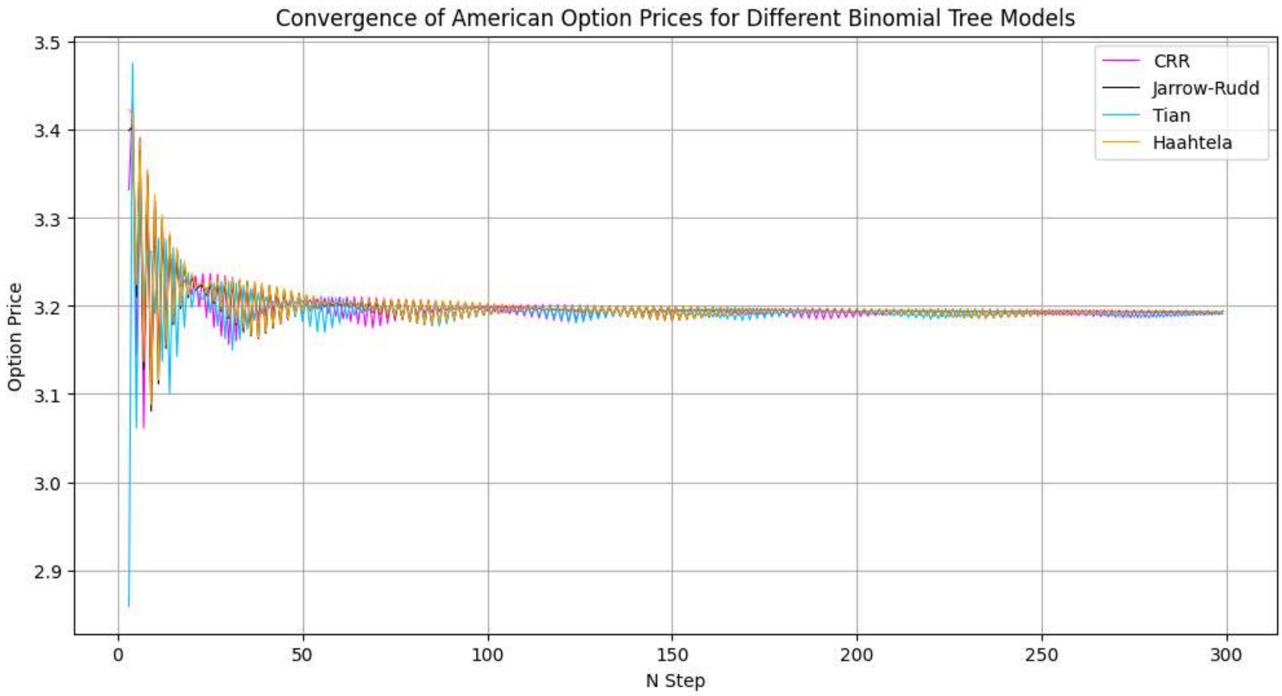

Figure 9: Convergence of American Option Prices for Different Binomial Tree Models

5) Convertible bond valuation and sensitivity analysis

The previous paragraph explored the versatility of alternative binomial tree models in convertible bond pricing, highlighting how variants such as the Tian, Jarrow-Rudd and Haahtela models address specific market dynamics and offer potential enhancements over the classic Cox-Ross-Rubinstein scheme.
These models improve pricing accuracy by capturing distinctions in the behaviour of the underlying asset, including skewness, volatility structure and other real-world characteristics that a standard binomial tree might overlook.
We conducted an in-depth analysis of alternative binomial trees, comparing them with traditional European and American options and verifying their proper convergence to the expected theoretical value. This analysis served as an important validation of the alternative approaches, demonstrating that models such as Jarrow-Rudd, Tian and Haahtela not only preserve theoretical consistency, but they are also capable of converging to reliable results within the context of fair value evaluation.
By analysing the structural differences of these models and conducting convergence analysis, we highlighted their potential for improving price accuracy, especially for complex instruments such as convertible bonds.
These improvements underline the flexibility and adaptability of binomial trees for modelling hybrid securities. Building on this foundation, in this section we move from theoretical considerations to practical implementation, focusing on the application of the CRR model and its alternatives to the valuation of convertible bonds.
Then, based on this theoretical validation, the present section focuses on the practical application of these alternative approaches in a more complex context: the valuation of convertible bonds. By leveraging the base structure of the code implemented for the traditional CRR model, we integrated the parameters u, d, and p specific to the alternative trees.
The objective is to address the same valuation problem previously solved with CRR but using the anticipated variants to explore how these models influence results in the context of hybrid instruments, such as convertible bonds.
Furthermore, in this section we introduce a comparative sensitivity analysis, assessing how the choice of the binomial model influences pricing outcomes and conversion probabilities. This analysis sheds light on the practical implications of model

choice, demonstrating the impact of factors such as volatility, interest rates and share price fluctuations on the valuation process.
Presenting the results of different binomial schemes, the section highlights the strengths and limitations of each approach, guiding practitioners in choosing the most appropriate model based on specific market conditions and bond characteristics.

Pricing Assessment with 5-steps

We now delve into the practical application of the Jarrow-Rudd, Tian and Haahtela models by analysing the Convertible Bond Price Binomial Tree and the Conversion Probability Binomial Tree. These visual tools not only depict the pricing structure but also illustrate the evolving decision-making process of bondholders at each node, highlighting the differences introduced by the alternative models.
The Convertible Bond Price Binomial Tree represents the recursive valuation process where the convertible bond value is determined at each node. At the terminal nodes, the bond value is derived by comparing the payoff of immediate conversion with the bond face value and accrued coupon payments.
Moving backward through the tree, intermediate nodes incorporate the discounted expected value of future payoffs, weighted by the risk-neutral probabilities. The results reveal how the unique assumptions of each alternative model influence the valuation process. The Jarrow-Rudd model, grounded in a symmetric framework, produces a straightforward, smooth progression of values across the tree. This regularity makes it an effective and computationally efficient alternative in stable market conditions. The Tian model, by accounting for skewness in the underlying stock price distribution, yields a wider range of values, particularly evident in extreme upward or downward scenarios. This skewness enables it to better reflect real-world asset behaviour, especially in volatile markets.
The Haahtela model, with its flexibility to move beyond the lognormal assumption, demonstrates the broadest range of values. This adaptability allows it to capture irregular market dynamics, making it particularly suited to environments with atypical price behaviours.
The bond price was evaluated, as shown in the trees below, using 5-steps.
The speed of convergence is a crucial metric in numerical methods, especially for financial modeling. It refers to how quickly a model approaches the theoretical fair value as the resolution of the tree increases. Faster convergence means that fewer steps are required to achieve a given level of accuracy.
In the context of convertible bond pricing, where valuation must account for both equity-like and bond-like features, a model with a high speed of convergence ensures that complex features such as conversion probabilities and early exercise decisions can be captured accurately without excessive computational overhead. This analysis provided an opportunity to examine the initial price structures generated by these models under minimal tree resolution, emphasizing their practical differences.
Our results showed that, even with only 5 steps, the alternative models delivered fair value estimates that were consistent with the CRR benchmark (see Figures 10, 11 and 12). Despite their distinct methodologies - such as Tian's skewness adjustment, Jarrow-Rudd's symmetric structure and Haahtela's flexible distribution - all models demonstrated reasonable accuracy, indicating their robustness. Our analysis also revealed how these structural differences influence the distribution of bond prices at the node level, providing insights into their potential strengths under varying market conditions.

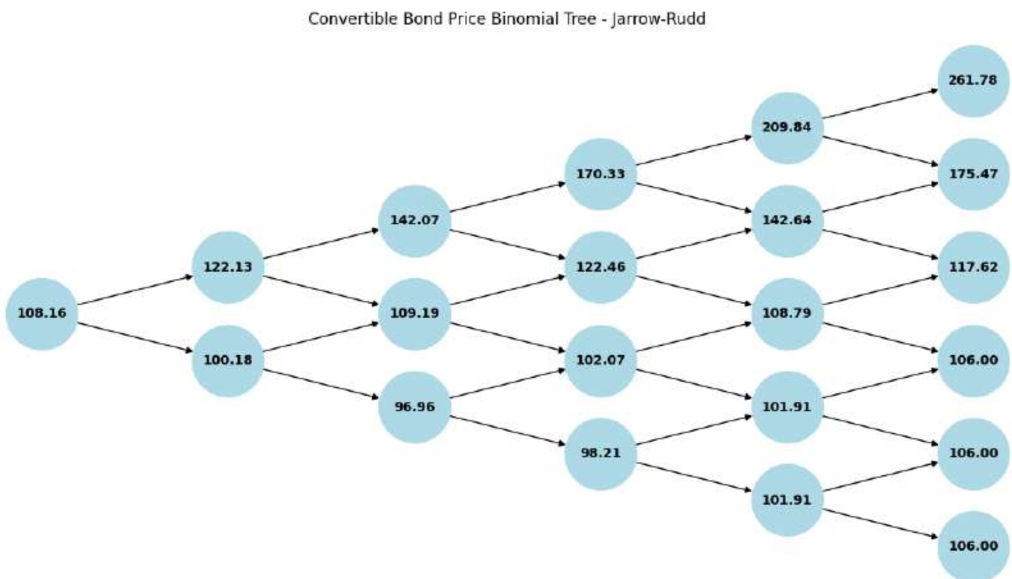

Figure 10a: 5-Steps Convertible Bond Price Binomial Tree – Jarrow Rudd Model

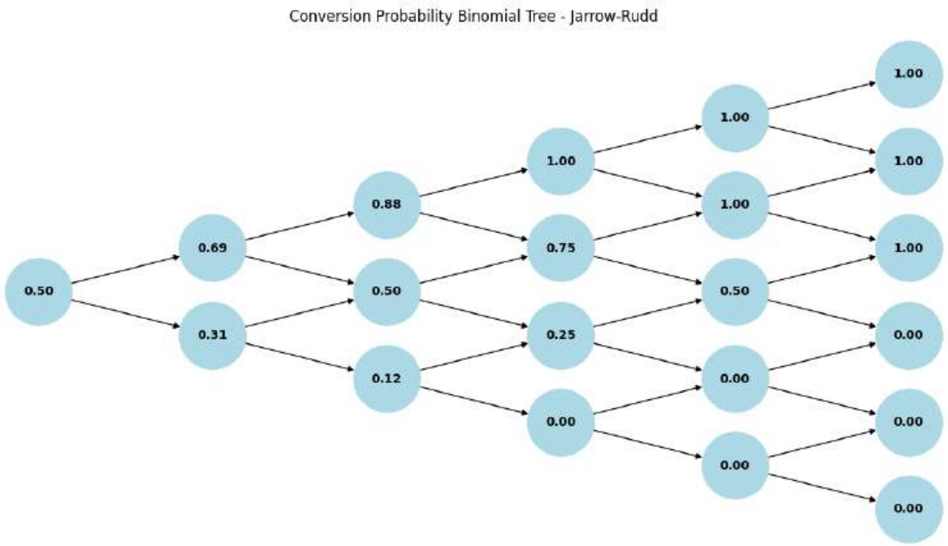

Figure 10b: 5-Steps Conversion Probability Binomial Tree – Jarrow Rudd Model

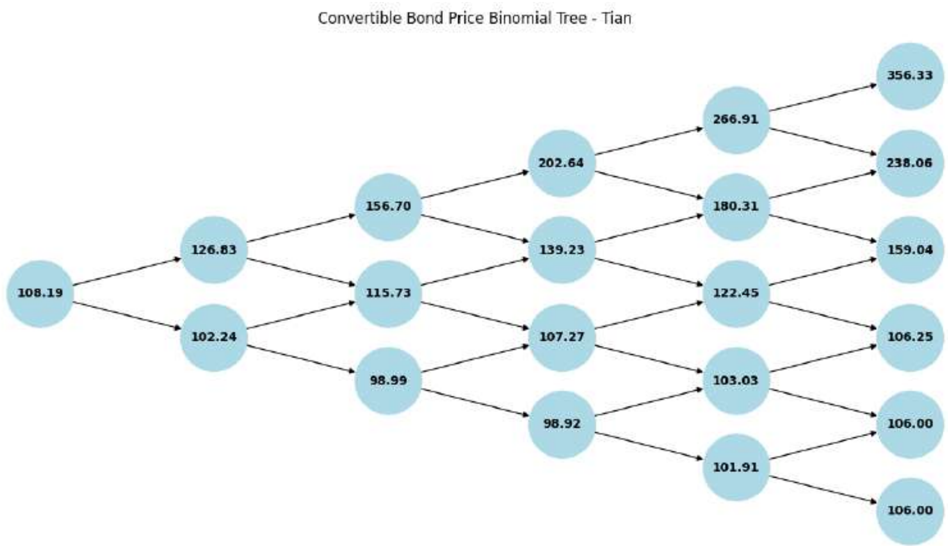

Figure 11a: 5-Steps Convertible Bond Price Binomial Tree – Tian Model

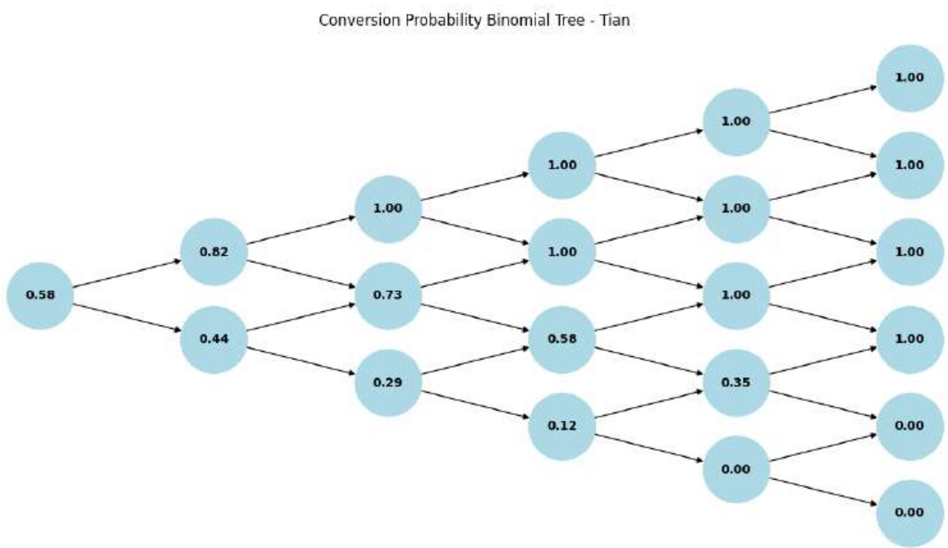

Figure 11b: 5-Steps Conversion Probability Binomial Tree – Tian Model

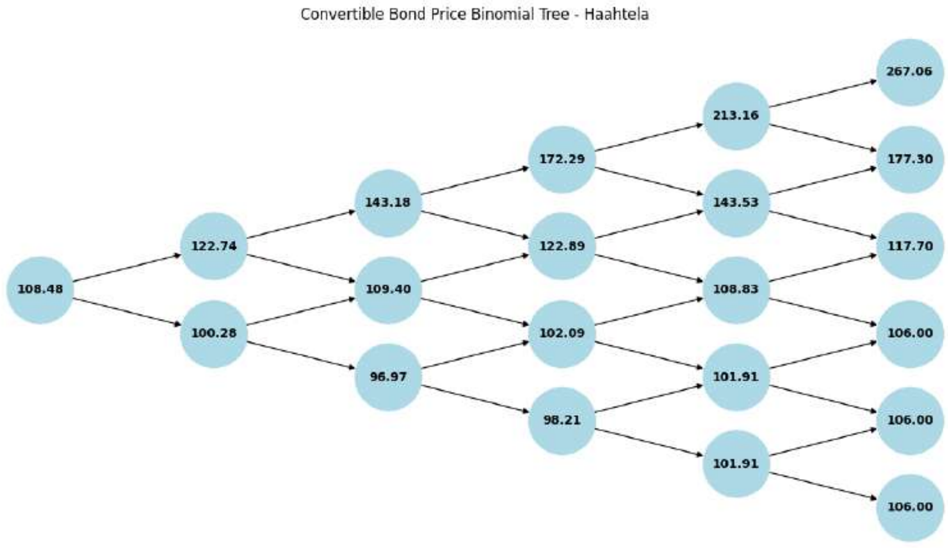

Figure 12a: 5-Steps Convertible Bond Price Binomial Tree – Haahtela Model

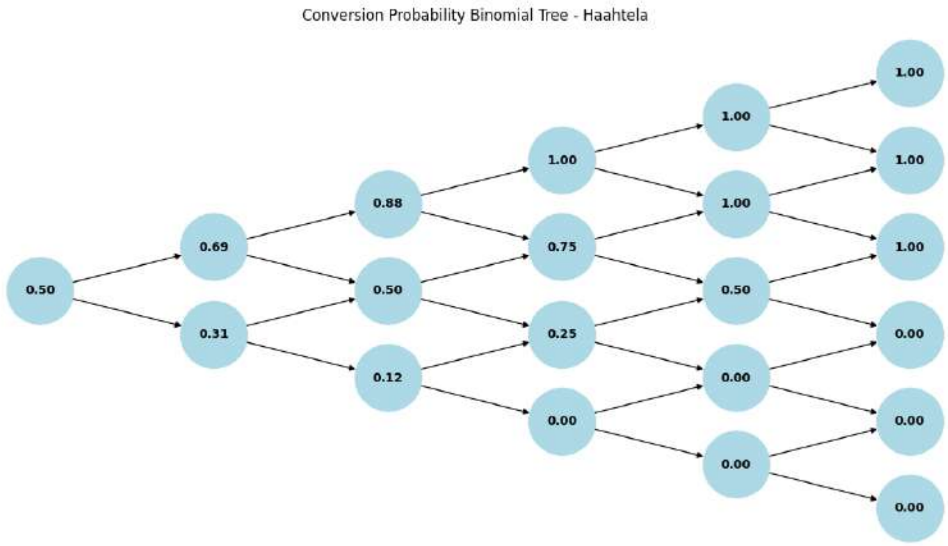

Figure 12b: 5-Steps Conversion Probability Binomial Tree – Haahtela Model

Convertible Bond Price Convergence of Alternative Binomial Trees

In this part, we expanded the analysis by increasing the number of steps in the binomial trees to 120, allowing for a comprehensive study of convergence behaviour.

In Figure 13, we provide a representation of the **convergence of alternative binomial models** in the valuation of convertible bonds. Examining the lines corresponding to the various approaches, it is evident that all models - Jarrow-Rudd, Tian and Haahtela - converge toward the same fair value limit, aligning with the results of the traditional CRR model. This observation is significant because it demonstrates that alternative trees, despite their structural differences, maintain theoretical consistency in the calculation of convertible bond values.

The convergence behaviour among the models suggests that these alternative approaches are as robust as the CRR model, with negligible differences in terms of convergence speed. Specifically, all models achieve stability in their values after a certain number of steps in the tree construction. The initial oscillations, visible mainly in the early steps of the graph, quickly diminish as the values converge toward a stable and reliable price.

This behaviour indicates that, in this specific case, the differences in the formulations of u, d, and p parameters do not significantly affect the model's ability to deliver accurate valuations.

One key implication of our analysis is that alternative models are not only theoretically valid but also practical and useful in real-world applications. The comparability in convergence speed between alternative models and the CRR highlights that they can be implemented without compromising computational efficiency. Therefore, models such as Jarrow-Rudd, Tian or

Haahtela can be confidently used in practical applications, especially in scenarios where their unique characteristics provide additional advantages.

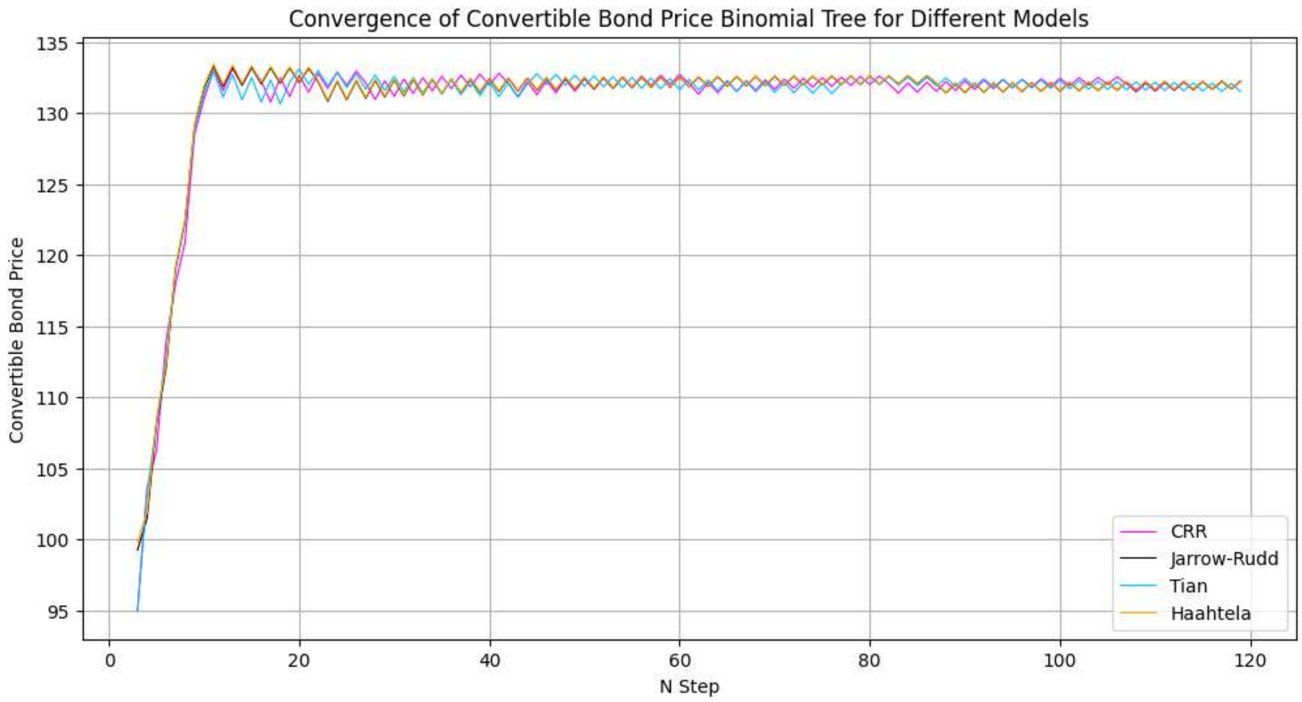

Figure 13: Convergence of Convertible Bond Price of the Alternative Binomial Trees

Moreover, it is interesting to observe that, in the specific case under consideration, the convergence of the models follows a similar pattern even in finer details (see Figure 13). For example, although the Tian model introduces the management of skewness in price distributions and the Haahtela model offers flexibility beyond lognormal assumptions, both exhibit a convergence behaviour that is almost indistinguishable from that of the other models. This result reinforces the idea that the choice between models can be guided not so much by their ability to converge - which is guaranteed - but rather by the specific market requirements or input conditions.

Another significant aspect concerns the interpretation of the plot's stable and aligned behaviour. It demonstrates that regardless of the model chosen, consistent results can be achieved in determining the fair value of the convertible bond. This is particularly important for complex instruments like convertible bonds, which require models capable of handling both equity-related characteristics (conversion option) and bond-related features (coupons and nominal value) simultaneously.

In short, the graph provides solid confirmation of the robustness of alternative approaches, highlighting their ability to converge to the expected theoretical value with a convergence speed comparable to that of the CRR model. This observation makes the alternative models not only a valid choice for convertible bond valuation but also flexible tools that can be employed to address a variety of market scenarios.

In the specific case we analysed, the behaviour of the models confirms that, despite introducing theoretical and structural differences, the alternative trees maintain a level of reliability and stability comparable to the CRR, making them practical and well-suited for financial decision-making processes.

Sensitivity Analysis

In this section, we analyse the sensitivities, known as "Greeks", which are key risk measures in option pricing, that quantify how the value of an option responds to changes in various market factors. These sensitivities are usually calculated as partial derivatives of the Black-Scholes-Merton (BSM) formula. These partial derivatives show the option price change to a small change in the parameters of the formula.

The Greeks include Delta, Gamma, Vega, Theta, and Rho, allowing us to break down complex price movements into measurable components. This analysis is essential for effective hedging strategies, risk management, and for understanding the behaviour of options under various market conditions.

In numerical methods, like the Cox-Ross-Rubinstein binomial tree model, Greeks are approximated using finite-difference techniques.

Unlike analytical approaches, these numerical methods involve discretization, where the underlying asset price path is divided into a finite number of steps. The lack of a closed-form solution means Greeks must be estimated by bumping input parameters and observing the resulting changes in the bond price.

Delta $(\mathbf{\Delta})$ is calculated by changing the underlying asset price incrementally:

$$\Delta \approx \frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S} \quad (16)$$

Gamma $(\mathbf{\Gamma})$ requires two levels of bumping to assess the change in Delta:

$$\Gamma \approx \frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{\Delta S^2} \quad (17)$$

Vega $(\boldsymbol{\nu})$ approximates sensitivity to volatility:

$$\vartheta \approx \frac{f(\sigma+\Delta\sigma)-f(\sigma-\Delta\sigma)}{2\Delta\sigma} \quad (18)$$

In addition to the computation of the Greeks, we conducted a further validation to confirm that the computation of the CRR has been conducted correctly.
The inclusion of Taylor approximations further validates the reliability of the binomial tree models, offering a quick and efficient method to estimate.
The Taylor series expansion provides a framework to approximate the change in the convertible bond price as a function of multiple variables, such as the underlying asset price, volatility and time. Using a first-order Taylor approximation, the convertible bond price change is expressed as a linear combination of partial derivatives (Greeks), while the inclusion of second-order terms, such as Gamma, accounts for curvature effects.
Then, given that the Greeks of financial derivatives are computed, we performed an approximate valuation of the instrument. By utilizing the first-order derivatives of the option price with respect to the relevant risk parameters, we applied the general Taylor formula in this context.
The larger the shock applied to the reference parameter, the less accurate is the approximation.
Comparing these approximations with the exact prices calculated for each model highlights the alignment between the numerical sensitivities and the bond price behaviour (see Table 1).

Greeks	CRR	Tian	Jarrow-Rudd	Haahtela
Delta	0.4076	0.4176	0.429	0.4291
Gamma	0.001	-0.001	-0.001	0.001
Vega	69.7186	43.1465	62.4383	62.5352

Table 1: Results of the Greek sensitivities

For the CRR model, the Taylor approximations demonstrate good agreement with exact prices across all Greeks, with minor deviations observed for Δ and ϑ, likely reflecting slight non-linearities in the stock price and volatility sensitivities.
The Taylor estimate of 131.765 for Δ closely approximates the exact price of 131.967.
The Tian model reveals similar trends but shows a slightly different sensitivity structure. For Δ, the Taylor approximation 131.981 again tracks the exact price 132.188 well, though some deviations emerge for ϑ, reflecting model-specific nuances in interest rate and volatility sensitivities.
In the Jarrow-Rudd model, the Taylor approximation for Δ of 132.204 again aligns well with the exact price 132.416, but the ϑ sensitivity shows reduced deviations compared to the CRR model.
Finally, the Haahtela model performs similarly to Jarrow-Rudd, with a high degree of alignment between Taylor approximations and exact prices. For instance, Δ and Γ produce approximations within a tight range of the exact values, while ϑ again shows minor discrepancies due to higher-order effects. These comparisons illustrate that the Taylor approximation acts as a diagnostic checkpoint, confirming the validity of Greek calculations and the numerical implementation of each binomial tree model. Moreover, the observed differences between the approximations and the exact prices highlight the unique sensitivities of each model, offering deeper insights into their structural characteristics and applicability to convertible bond pricing. This further validates the convergence properties explored earlier, as Taylor approximations consistently approach exact prices as the binomial tree models stabilize with increasing time steps. By integrating analytical and numerical methods, the Taylor approach reinforces confidence in the robustness and accuracy of the alternative models studied.

6) Market Case Study

This section presents an experiment in pricing Convertible bonds having different shares as the underlying to the conversion option. We considered the forty stocks in the German DAX index because currently the most active market for these hybrid

instruments is Germany (source: Bloomberg®, module: Fixed Income Search). The reference date for the valuations is December 31, 2024.
The EOY (End-Of-Year) closing prices together with the description of the underlyings are shown in Table B.1 in the Appendix. These constitute the spot prices, S, of the model.
As for the estimation of dividend yields, we employed the term structures of continuous dividend yield $q(t)$ implied by the call-put parity or by forward contracts quotations. If no implied values are contributed, due to the absence of actively traded derivatives written on an underlying, the value considered remains constant for all maturities and is set equal to the ratio of the cash dividend paid, divided by the spot, as is traditionally defined in corporate finance. Table B.2 in Appendix shows these estimates for eight different maturities (6M, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y, and 10Y) for all forty German stocks.
The implied volatilities are lognormal, $\sigma(K,t)$, and they are calculated from three different strike prices (moneyness: 80%, 100% and 120%) and eight different maturities (6M, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y and 10Y).
A two-dimensional linear interpolation (strike price - time) is conducted in the pricing routines in order to choose the most suitable volatility for the valuation.
If there are no vanilla European options, the volatility surface is flat, characterized by a single value of volatility estimated historically by the close-to-close method. Table B.3 shows the three sections of moneyness for all German stocks considered in the experiment.
The risk-free term structure we used for the analysis has one day as tenor. The ESTER curve is shown in Figure B.1 in the Appendix. Under the assumption that the issuer of the convertible bond is the same as the stock, to take into account the correct creditworthiness, we rely on CDS premiums traded in the market. If this information is not available due to the absence of active quotes, a single value computed from the Z-spread closest to the maximum maturity considered in the experiment, i.e., 10 years, is used instead of the entire CDS term structure. The CDS curves are shown in Table B.1.
To estimate the impact on price and the main measures of option sensitivity, namely Delta Δ, Gamma Γ and Vega ϑ, we assumed different convertible bonds for the different alternative binomial trees discussed. These scenarios are designed to test whether the specific results reported in Table 1 are confirmed with different market scenarios.
We now describe the financial characteristics common to all hybrid instruments considered in the different scenarios: the coupon rate is 3% paid semi-annually, the face amount of the bond is 100, the discretization interval of the numerical scheme is one day, $\Delta t = 1/360$. The type of exercise considered for the option is American, i.e., the right can be exercised at any instant in time.
Here are the parameters which define the scenarios for all 40 stocks:

- A structured product maturity from six months to 10 years in six-month steps.

- A moneyness of the option ranging from 50% to 150% with a 10% step.

All interpolations of market term structures are made by dynamically considering the analyzed maturities/strikes.
These simulations are conducted for all four binomial approaches considered (CRR, Jarrow-Rudd, Tian and Haahtela) and for the quantitative measures of price, Delta, Gamma and Vega.
Thus, we produced 140,800 scenarios, stored in tensors of 5 dimensions each: 40 stocks ×20 maturities ×11 strike prices×4 trees×4 quantitative measures.
In terms of price, the values of the alternative binomial trees produced extremely aligned results; in fact, the maximum valuation gap for all scenarios is no more than one cent. Regarding the calculated CRR Greeks, Jarrow-Rudd and Haahtela produced aligned values, while the Tian approach again showed different sensitivity with respect to volatility (ϑ) especially in experiments where the convertible bond had a longer maturity and an ATM strike. The first- and second-order sensitivities with respect to spot (Δ and Γ) showed alignment with those estimated with the other numerical approaches. The Matlab code written for these experiments is available upon request.

7) Conclusions

This study set out to explore and evaluate the applicability of alternative stochastic binomial tree models in the valuation and risk analysis of convertible bonds, a complex class of hybrid securities that combine features of both equity and debt instruments. While the Cox-Ross-Rubinstein binomial tree remains the most widely adopted method in practice and literature for this purpose, we identified a gap in research concerning the implementation and effectiveness of alternative binomial models in this context.
To address this, we systematically examined three well-established binomial models (Haahtela, Jarrow-Rudd and Tian) and compared them against the traditional CRR approach. We began by analyzing the theoretical foundations and structural differences of each model, particularly their assumptions regarding drift, volatility, and node recombination. These differences can have meaningful implications for pricing accuracy and computational behavior, especially when applied to the valuation of instruments with embedded optionality such as convertible bonds.

As a preparatory step, we implemented each model in the pricing of standard European and American options. This served both as a validation of the correct numerical implementation and as a benchmark for convergence behavior. All alternative models demonstrated satisfactory convergence to theoretical option values, thereby confirming their numerical soundness and suitability for more advanced applications.

We then extended the analysis to convertible bonds, adapting the CRR-based lattice framework to each alternative method. This involved step-by-step reconstruction of the pricing trees. Sensitivity analyses were conducted to measure the responsiveness of each model to changes in key risk parameters: Delta, Gamma and Vega. While Delta and Gamma sensitivities remained broadly consistent across models, the Jarrow-Rudd tree exhibited a noticeably different Vega profile, suggesting that its underlying assumptions about return distributions may influence how volatility risk is captured.

Finally, we validated the real-world applicability of these models through a series of empirical tests using market data. These case studies confirmed that the alternative binomial trees not only provide robust estimates of convertible bond values but also yield consistent risk metrics across varying market conditions. This reinforces the view that, when correctly implemented, these models can serve as reliable tools in both academic research and practical financial engineering.

In addition to serving as a comparative study, this research contributes a practical framework for implementing, testing, and validating alternative lattice methods in the pricing of convertible bonds. The structured approach provides a replicable path for practitioners and researchers aiming to extend valuation models beyond conventional techniques.

Looking forward, a natural continuation of this research involves extending the analysis to trinomial stochastic trees, which offer greater flexibility and can better capture skewness, kurtosis, and complex early-exercise features in convertible bond contracts. Adapting both traditional and alternative trinomial approaches - such as the Boyle and Tian trinomial models - using the same rigorous methodology outlined in this study could provide further insight into the comparative strengths of different lattice-based techniques in modeling hybrid financial instruments.

Ultimately, this work enhances the quantitative toolkit available for convertible bond valuation and risk assessment, offering more nuanced and potentially more accurate modeling alternatives that can adapt to various market scenarios and investor requirements.

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Appendix A – Cox Ross Rubinstein Tree for option pricing

The Cox-Ross-Rubinstein (CRR) binomial model provides a discrete-time approximation of the dynamics of asset prices and serves as a foundational tool for derivative pricing. A crucial aspect of the model construction lies in the derivation of its parameters: the up and down factors (*u* and *d*) and the risk-neutral probability (*p*).
These parameters must be chosen to ensure no-arbitrage conditions and to allow convergence of the binomial model to the continuous-time Black-Scholes model as the number of time steps increases (see Figure A.1). This section presents a formal derivation of the CRR parameters, grounded in the principles of risk-neutral valuation and probabilistic convergence.

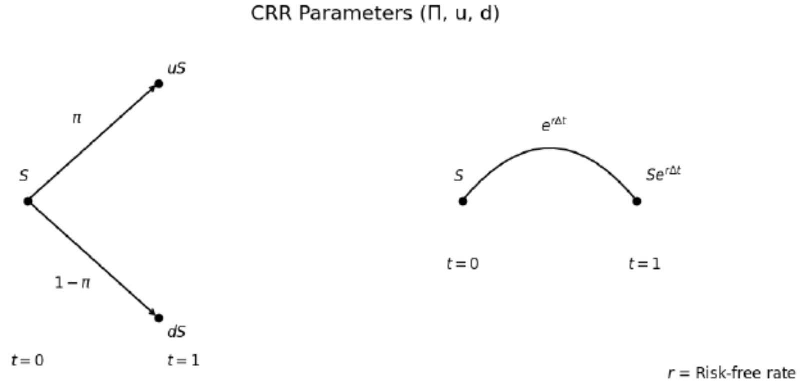


Figure A.1: A graphical representation of CRR parameters

The Black-Scholes analytical formulas (BS closed formulas) cannot provide accurate valuations for all types of options traded in financial markets. They are unable to fairly value options with non-standard features, such as those that allow early exercise (Bermuda/American options) or have complex payoffs (exotic options). In such cases, numerical methods must be used to value the derivative.
The literature offers numerous mathematical techniques that align with the principles of the Black-Scholes framework. We discuss and implement one of the stochastic binomial trees which is most used by quantitative analysts: the CRR model.
To align with the stochastic dynamics assumed by the Black-Scholes framework, Cox, Ross, and Rubinstein proposed selecting the parameters *u* and *d* such that, for each time interval Δt the projected future asset values match the theoretical mean and variance of the continuous model (Cox, Ross and Rubinstein, 1979). Assuming a risk-neutral environment, the expected rate of return of the stock is equal to the risk-free interest rate r. Therefore, the expected stock price at the end of interval Δt is $S \cdot e^{r\Delta t}$ which is the stock price at the beginning of the interval. This leads to match the expected value of the asset in the binomial model to the one implied by the BSM model.

First moment matching $\mathbb{E}(S)$:

$$S \cdot e^{r\Delta t} = \Pi\, uS + (1-\Pi)dS \quad \text{(A.1)}$$

Dividing both sides by S:

$$e^{r\Delta t} = \Pi u + d - \Pi d$$

$$\Pi(u-d) = e^{r\Delta t} - d \rightarrow \Pi = \frac{e^{r\Delta t} - d}{u-d} \quad \text{(A.2)}$$

The stochastic process assumed by the GBS framework (i.e. a Geometric Brownian motion) implies that the variance (VAR) of its rate of change in a short interval of length Δt is $\sigma^2 \Delta t$.
Since the variance of a random variable S is defined as $E(S)^2 - E(S)^2$, where $E(\cdot)$ represents the expected value, we can derive the second equation that connects the second moment of the stochastic process to the dynamics of the binomial tree.

Second moment matching: $VAR(S) = \mathbb{E}(S^2) - \mathbb{E}(S)^2$ (A.3)

$$\underbrace{\Pi\, u^2 + (1-\Pi)d^2}_{\mathbb{E}(S^2)} - \underbrace{[\Pi\, u + (1-\Pi)d]^2}_{\mathbb{E}(S)^2} = \sigma^2 \Delta t \quad \text{(A.4)}$$

True under BSM framework, where:

$$\sigma^2 \Delta t = \Pi\, u^2 + (1-\Pi)d^2 - \Pi^2 u^2 - 2\Pi(1-\Pi)ud - (1-\Pi)^2 d^2$$

$$\sigma^2 \Delta t = u^2(\Pi - \Pi^2) + [(1-\Pi) - (1-\Pi)^2]d^2 - 2\Pi(1-\Pi)ud$$
$$\sigma^2 \Delta t = u^2\Pi(1-\Pi) + (1-\Pi)[1-(1-\Pi)]d^2 - 2\Pi(1-\Pi)ud$$
$$\sigma^2 \Delta t = \Pi(1-\Pi)[u^2 - 2ud + d^2] = \Pi(1-\Pi)(u-d)^2 \quad \text{(A.5)}$$

Substituting Π from eq. (A.2):

$$\Pi(1-\Pi) = \Pi - \Pi^2 = \frac{e^{r\Delta t} - d}{u-d} - \frac{e^{2r\Delta t} - d + 2d \cdot e^{r\Delta t} - d}{(u-d)^2}$$
$$= \frac{e^{r\Delta t}u - ud - e^{r\Delta t}d + d^2 - e^{2r\Delta t} + 2d \cdot e^{r\Delta t} - d^2}{(u-d)^2}$$
$$= \frac{e^{r\Delta t}(u - d + 2d) - ud - e^{2r\Delta t}}{(u-d)^2}$$
$$= \frac{e^{r\Delta t}(u + d) - ud - e^{2r\Delta t}}{(u-d)^2} \quad (A.6)$$

Then, solving for $\prod$ from the first moment equation and substituting this value into the second moment equation, we obtain:

$$\sigma^2 \Delta t = \frac{e^{r\Delta t}(u + d) - ud - e^{2r\Delta t}}{(u-d)^2}(u-d)^2$$
$$\sigma^2 \Delta t = e^{r\Delta t}(u + d) - ud - e^{2r\Delta t} \quad \text{(A.7)}$$

Recalling that Cox, Ross and Rubinstein assumed that $u = \frac{1}{d}$, we obtain a 3×3 system which allows to express the parameters Π, u, d in terms of $r, \sigma, \Delta t$:

$$\begin{cases} \Pi = \dfrac{e^{r\Delta t} - d}{u - d} & \rightarrow \alpha \\ \sigma^2 \Delta t = e^{r\Delta t}(u + d) - ud - e^{2r\Delta t} & \rightarrow \beta \\ u = \dfrac{1}{d} & \rightarrow \gamma \end{cases} \quad (A.8)$$

$$\gamma \rightarrow \beta$$

$$e^{r\Delta t}\left(u + \frac{1}{u}\right) - u\frac{1}{u} - e^{2r\Delta t} = \sigma^2 \Delta t$$
$$u + \frac{1}{u} = \frac{\sigma^2 \Delta t + 1 + e^{2r\Delta t}}{e^{r\Delta t}}$$
$$u + \frac{1}{u} = e^{-r\Delta t}[\sigma^2 \Delta t + 1 + e^{2r\Delta t}]$$
$$u + \frac{1}{u} = e^{-r\Delta t}\sigma^2 \Delta t + e^{-r\Delta t} + e^{r\Delta t} \quad (A.9)$$

Under the hypothesis that Δt is very small:

$$e^{-r\Delta t} \approx (1 - r\Delta t), e^{+r\Delta t} \approx (1 + r\Delta t), r\sigma^2 \Delta t^2 \rightarrow 0 \quad (A.10)$$

Thus, the quadratic equation becomes:

$$u + \frac{1}{u} \approx (1 - r\Delta t)\sigma^2 \Delta t + 1 - r\Delta t + 1 + r\Delta t =$$
$$= \sigma^2 \Delta t - r\sigma^2 \Delta t^2 + 2 = \sigma^2 \Delta t + 2$$
$$u + \frac{1}{u} = \sigma^2 \Delta t + 2 \rightarrow u^2 + 1 = \sigma^2 u \Delta t + 2u.$$
$$u^2 - (\sigma^2 \Delta t + 2)u + 1 = 0 \quad (A.11)$$

Solving for u:

$$u = \frac{\sigma^2 \Delta t + 2 \pm \sqrt{(\sigma^2 \Delta t + 2)^2 - 4}}{2} = \frac{\sigma^2 \Delta t + 2 \pm \sqrt{\sigma^4 \Delta t^2 + 4\sigma^2 \Delta t + 4 - 4}}{2}$$
$$= \frac{\sigma^2 \Delta t + 2 \pm \sqrt{\sigma^4 \Delta t^2 + 4\sigma^2 \Delta t}}{2} = \frac{\sigma^2 \Delta t}{2} + 1 \pm \sigma\sqrt{\Delta t} \quad (A.12)$$

For a very small Δt, $\sigma^2 \Delta t^2$ tends to 0. Since $\sqrt{\Delta t}$ is far larger than Δt for small Δt, and σ^2 is relatively smaller than σ, we can ignore the first term $\frac{\sigma^2 \Delta t}{2}$.

$$u \approx 1 \pm \sigma\sqrt{\Delta t} \quad \text{(A.13)}$$
$$u \approx \exp(\sigma\sqrt{\Delta t}) \text{ because } u > 1 \Longrightarrow u \neq \exp(-\sigma\sqrt{\Delta t})$$

Hence, the below set of parameters enables the construction of a binomial stochastic tree that fully aligns with the Black-Scholes pricing framework. Thus, if the number of time intervals approaches infinity $N \to \infty$, the model theoretically converges to the closed-form valuation formula for European vanilla options.

$$\begin{cases} e^{r\Delta t} = \prod u + (1 - \prod)d \\ e^{r\Delta t}(u + d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t \\ u = \frac{1}{d} \end{cases} \rightarrow \begin{cases} \prod = \frac{e^{r\Delta t} - d}{u - d} \\ u = e^{\sigma\sqrt{\Delta t}} \\ d = e^{-\sigma\sqrt{\Delta t}} \end{cases} \quad \text{(A.14)}$$

The numerical formulas of the binomial approach can be extended to more underlyings by introducing the parameter b called cost-of-carry (Haug, 2007).
Based on the value of the parameter b, we reach a pricing framework applicable to a wide range of underlying assets for which call or put options can be written. The adjustment needed is in the definition of the risk-neutral probability Π, equation (A.2).

If $b = r$ the definition is suitable to be used for the pricing of options written on shares that pay no dividend.

If $b = r - q$ the definition is suitable to be used for the pricing of options written on shares/indexes with a continuous dividend yield q.

If $b = 0$ the definition is suitable to be used for the pricing of options on futures.

If $b = r - r_{FOR}$ the definition is suitable to be used for the pricing of currency options.

Binomial Option Pricing: The Cox-Ross-Rubinstein

Building on the derivation of equation (A.14) in the previous section, which defines the up and down factors and risk-neutral probabilities aligning with the Geometric Brownian Motion framework, this section extends these principles to practical applications. It explores the implementation of the CRR binomial tree model to price European and American options, emphasizing its adaptability for different payoff structures. This transition from theoretical parameter derivation to numerical application demonstrates that the CRR model bridges discrete and continuous approaches to option valuation, providing a robust framework for pricing various financial derivatives.
The Binomial Option Pricing is a numerical method for pricing options and derivative securities. Unlike analytical solutions, this numerical method is versatile and can handle a broader range of options for which no closed-form solutions exist.
The binomial method is the most widely used numerical approach for pricing American options on stocks, futures, and currencies. Originally developed by Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979), this method approximates Geometric Brownian Motion with a recombining binomial tree.
When the number of time steps is large, the binomial tree converges to the continuous Black-Scholes-Merton model for European options. The binomial model is especially suited for pricing American options, where no closed-form solution exists, and for many exotic options.
In a binomial tree, the asset price can increase by a factor u with probability p or decrease by a factor d with probability $(1 - \prod)$ over each time step Δt. The number of time steps is n. Each node is represented by (j, i) where j is the number of time steps to a node in the tree, and i represents the number of upward moves.
The first node $(j = 0, i = 0)$ of the tree progresses with each step. If the asset price goes up at the second node, it will be assigned $(j = 1, i = 1)$. If the asset price goes down at the first-time step, we have $(j = 1, i = 0)$, as shown in Figure A.2.
The number of paths leading to a node (j, i) is $\frac{j!}{i!(j-i)!}$, and the equivalent probability of reaching node (j, i) is $\frac{j!}{i!(j-i)!}\prod^i(1 - \prod)^{j-i}$ (Giribone, 2024).

To price European call or put options, we only need the end nodes at time n, such that (with X denoting the strike price):

$$c = e^{-rT}\sum_{i=0}^{n}\left(\frac{n!}{i!\,(n-i)!}\right)\prod^i(1 - \prod)^{n-i}\max[Su^i d^{n-i} - X, 0] \quad \text{(A.15)}$$

$$p = e^{-rT}\sum_{i=0}^{n}\left(\frac{n!}{i!\,(n-i)!}\right)\prod^i(1 - \prod)^{n-i}\max[X - Su^i d^{n-i}, 0] \quad \text{(A.16)}$$

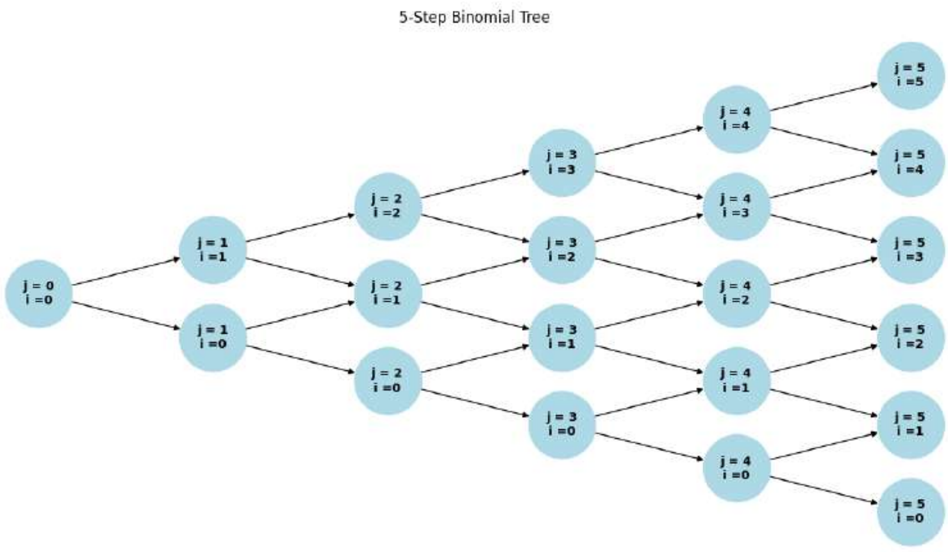


Figure A.2: Standard indexing for a 5-Step Binomial Tree

Many nodes will be out-of-the-money, so instead of starting the count from the lowest node $(i = 0)$, we can improve the algorithm efficiency by beginning at a (for a call option), which is the smallest non-negative integer greater than $\frac{\ln\left(\frac{X}{Sd^n}\right)}{\ln\left(\frac{u}{d}\right)}$. This gives:

$$c = e^{-rT} \sum_{i=a}^{n} \left(\frac{n!}{i!\,(n-i)!}\right) \prod^{i} (1 - \prod)^{n-i} (Su^i d^{n-i} - X) \quad (A.17)$$

$$p = e^{-rT} \sum_{i=a}^{n} \left(\frac{n!}{i!\,(n-i)!}\right) \prod^{i} (1 - \prod)^{n-i} (X - Su^i d^{n-i}) \quad (A.18)$$

Generalized European Binomial

The European binomial model in its more general form is:

$$c = e^{-rT} \sum_{i=0}^{n} \left(\frac{n!}{i!\,(n-i)!}\right) \prod^{i} (1 - \prod)^{n-i} g[S(T), X] \quad (A.19)$$

$$p = e^{-rT} \sum_{i=0}^{n} \left(\frac{n!}{i!\,(n-i)!}\right) \prod^{i} (1 - \prod)^{n-i} g[S(T), X] \quad (A.20)$$

where $S(T) = Su^i d^{n-i}$ and $g[S(T), X]$ represents any specified payoff function at maturity.
This highlights the versatility of the simple binomial model, as it can price any European option on a single asset with a payoff that is not path dependent.
For example, to determine the value of a power option with a payoff of $\max[S^2 - X, 0]$ at maturity, we simply replace $g[S(T), X]$ with $\max[(Su^i d^{n-i})^2 - X, 0]$. The variable *z* equals 1 if the contract is a call, and -1 if it is a put.

Cox-Ross-Rubinstein American Binomial Tree

Here, we examine how to apply the Cox-Ross-Rubinstein binomial tree to value American-style options. At each node, the asset price is given by:

$$Su^i d^{n-i}, \qquad with\ i = 0, 1, \ldots, j \quad (A.21)$$

where u and d are the up and down jump factors for each time interval $\Delta t = \frac{T}{n}$, with n being the number of time steps, as previously defined (see Figure A.3). The probability of the stock price increasing by factor u is given by equation (A.17).
Since the probabilities must add up to one, the probability of the stock price decreasing by d is $(1 - \prod)$.

The up and down factors and probabilities are selected to match the first two moments of the stock price distribution, ensuring that as Δt approaches zero, the probability distribution generated by the binomial tree converges to Geometric Brownian motion. A key benefit of this numerical pricing technique consists in its ability to compute the fair value of options with the possibility of early exercise, such as American and Bermudan options. This flexibility, however, introduces additional complexity compared to standard European-style valuation. Rather than simply calculating the payoff at maturity and propagating the value backwards using a conventional algorithm, it becomes necessary to evaluate, at each time step, whether exercising the option immediately yields a higher value than holding it. Consequently, at every node in the binomial tree, the option value is determined as the maximum between its immediate exercise payoff and its continuation value:

$$C_t = \max[C_{dead}; C_{alive}] = \max\left[S_t - K; \frac{C_u \cdot \Pi + C_d \cdot (1-\Pi)}{1+R}\right] \quad (A.22)$$

$$P_t = \max[P_{dead}; P_{alive}] = \max\left[K - S_t; \frac{P_u \cdot \Pi + P_d \cdot (1-\Pi)}{1+R}\right] \quad (A.23)$$

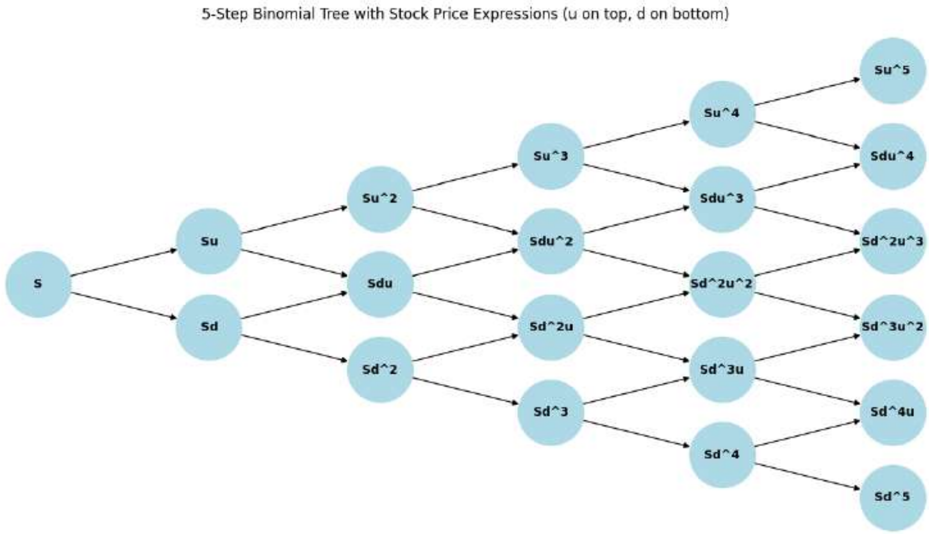


Figure A.3: 5-Step Binomial Tree with stock price movements

Appendix B – Market Inputs

This appendix reports the market data used for the numerical experiment of pricing and sensitivity estimation with the different alternative binomial models. Market values are downloaded from Bloomberg® at closing market values on December 31, 2024.

Underlyings			CDS							
Ticker	Name	Price	0.5	1	2	3	4	5	7	10
ENR GY Equity	Siemens Energy AG	50.38	9.4	13.7	19.4	25.2	29.6	31	44	64.5
SY1 GY Equity	Symrise AG	102.65	36	36	36	36	36	36	36	36
PAH3 GY Equity	Porsche Automobil Holding SE	36.35	48.5	53.1	68.5	83	110.6	146.35	182.1	218.9
MTX GY Equity	MTU Aero Engines AG	322	49	49	49	49	49	49	49	49
RHM GY Equity	Rheinmetall AG	614.6	50.4	58.2	75.6	96.3	117.1	137.1	166.2	185.3
DTG GY Equity	Daimler Truck Holding AG	36.85	97	97	97	97	97	97	97	97
SHL GY Equity	Siemens Healthineers AG	51.2	9.4	13.7	19.4	25.2	29.6	31	44	64.5
ZAL GY Equity	Zalando SE	32.39	71	71	71	71	71	71	71	71
QIA GY Equity	QIAGEN NV	43.045	44.65	44.65	44.65	44.65	44.65	44.65	44.65	44.65
SRT3 GY Equity	Sartorius AG	215.2	47	47	47	47	47	47	47	47
BNR GY Equity	Brenntag SE	57.88	63	63	63	63	63	63	63	63
AIR GY Equity	Airbus SE	154.46	20	20	27	34	41	48	68.5	92.5
ALV GY Equity	Allianz SE	295.9	9.05	11.5	14.6	20.5	26.36	32.5	43.6	53.6
RWE GY Equity	RWE AG	28.83	7.2	8.5	12.3	18	25.14	33.3	46.6	59.9
BAYN GY Equity	Bayer AG	19.314	34.1	39	52	68	86	103.5	135	170
BMW GY Equity	Bayerische Motoren Werke AG	78.98	16.8	19	26.9	34.7	48.1	61.5	84.5	108
CBK GY Equity	Commerzbank AG	15.725	18.6	23.2	30.2	36.6	44.785	53	69.3	90
DBK GY Equity	Deutsche Bank AG	16.64	13.7	16.9	28.8	39.1	49.73	61.5	82.3	102
BAS GY Equity	BASF SE	42.46	13.9	15.6	21.5	27.4	39.7	52	71	92
HEN3 GY Equity	Henkel AG & Co KGaA	84.7	6.8	8	10.6	13.3	17.1	21	29	39.5
SIE GY Equity	Siemens AG	188.56	9.4	13.7	19.4	25.2	29.6	31	44	64.5
VOW3 GY Equity	Volkswagen AG	89.04	35.2	39	50	66.3	89.2	112.5	152	183.5
EOAN GY Equity	E.ON SE	11.245	9.6	11	15.8	20.6	34.5	27.5	52.5	70.5
BEI GY Equity	Beiersdorf AG	124	14	14	14	14	14	14	14	14
HEI GY Equity	Heidelberg Materials AG	119.3	14.4	18.2	27.9	38.2	53.1	68	99	129
MUV2 GY Equity	Muenchener Rueckversicherungs-Gesellschaft AG in Muenchen	487.1	8.2	10.5	14.2	20.3	27.07	34.5	45.6	56.1
FRE GY Equity	Fresenius SE & Co KGaA	33.54	6.8	10.6	18.6	27.5	35.27	40	56.1	62.8
SAP GY Equity	SAP SE	236.3	26	26	26	26	26	26	26	26
MRK GY Equity	Merck KGaA	139.9	5.9	8.4	11.7	16.6	22.88	30	40.7	52.5
ADS GY Equity	adidas AG	236.8	24.3	27.1	31.9	37.8	45.19	52.9	65.2	78.7
DTE GY Equity	Deutsche Telekom AG	28.89	13.7	16.7	22	27.2	32.6	39.5	50.5	69.5
DHL GY Equity	Deutsche Post AG	33.98	7.5	9.1	12.5	15.5	20.9	26	36.5	46.5
FME GY Equity	Fresenius Medical Care AG	44.16	6.8	10.6	18.6	27.5	35.27	40	56.1	62.8
MBG GY Equity	Mercedes-Benz Group AG	53.8	17.2	19.9	25.3	34.4	47.5	60.5	83.5	107.5
IFX GY Equity	Infineon Technologies AG	31.4	72.7	78.7	87.1	95.3	101.73	107.3	116.1	125.1
DB1 GY Equity	Deutsche Boerse AG	222.4	18.4	20	22.2	25.6	29.75	34.3	42.2	49.9
VNA GY Equity	Vonovia SE	29.32	107	107	107	107	107	107	107	107
P911 GY Equity	Dr Ing hc F Porsche AG	58.42	67	67	67	67	67	67	67	67
HNR1 GY Equity	Hannover Rueck SE	241.4	8.4	10.9	14.7	21.4	27.79	35	48	58.5
CON GY Equity	Continental AG	64.82	13.6	16.4	31.2	46	65.7	85.5	122.5	146.5

Table B.1 Close prices for the Equity shares in EUR and CDS premiums for the Issuers in bps. Reference Date: EOY 2024

T/Equity	ENR GY Equity	SY1 GY Equity	PAH3 GY Equity	MTX GY Equity	RHM GY Equity	DTG GY Equity	SHL GY Equity	ZAL GY Equity	QIA GY Equity	SRT3 GY Equity
0.5	0	2.718	5.998	0	0.546	7.394	2.907	0	0.556	1.531
1	0	1.355	2.991	0	0.272	3.687	1.449	0	0.277	0.763
2	0.94	0.968	4.385	0.614	0.6	3.332	1.722	0.108	0.277	0.837
3	1.156	0.837	4.581	0.592	0.714	3.183	1.71	0.131	0.269	0.859
4	1.257	0.77	4.608	0.647	0.767	3.084	1.704	0.15	0.276	0.868
5	1.313	0.73	4.576	0.679	0.797	3.009	1.689	0.158	0.276	0.872
7	1.369	0.682	4.435	0.712	0.827	2.887	1.659	0.165	0.272	0.872
10	1.386	0.643	4.163	0.731	0.842	2.733	1.616	0.174	0.274	0.865

T/Equity	BNR GY Equity	AIR GY Equity	ALV GY Equity	RWE GY Equity	BAYN GY Equity	BMW GY Equity	CBK GY Equity	DBK GY Equity	BAS GY Equity	HEN3 GY Equity
0.5	3.898	1.68	7.583	7.339	0.212	8.687	6.065	5.357	6.65	4.64
1	1.944	1.68	3.781	3.659	0.106	4.332	3.024	2.671	3.316	2.314
2	2.214	1.68	3.619	2.977	0.354	3.896	2.697	3.36	3.626	2.35
3	2.284	1.68	3.526	2.717	0.434	4.503	2.556	3.558	3.673	2.358
4	2.301	1.68	3.449	2.573	0.473	4.715	2.47	3.595	3.655	2.356
5	2.301	1.68	3.386	2.477	0.496	4.778	2.41	3.589	3.616	2.336
7	2.281	1.68	3.259	2.347	0.52	4.718	2.323	3.517	3.514	2.289
10	2.215	1.68	3.08	2.21	0.535	4.464	2.211	3.357	3.335	2.21

T/Equity	SIE GY Equity	VOW3 GY Equity	EOAN GY Equity	BEI GY Equity	HEI GY Equity	MUV2 GY Equity	FRE GY Equity	SAP GY Equity	MRK GY Equity	ADS GY Equity
0.5	3.645	7.257	6.204	1.603	4.178	4.969	0	1.23	1.943	0.583
1	1.817	3.618	3.094	0.799	2.083	2.478	0	0.614	0.969	0.291
2	1.866	3.503	3.698	0.506	2.127	2.651	0	0.738	0.998	0.526
3	1.869	3.458	3.836	0.409	2.126	2.682	0	0.776	1.002	0.603
4	1.861	3.403	3.875	0.362	2.112	2.676	0	0.793	1	0.639
5	1.85	3.36	3.851	0.332	2.095	2.668	0	0.805	0.998	0.66
7	1.821	3.245	3.751	0.298	2.057	2.612	0	0.811	0.993	0.681
10	1.772	3.075	3.557	0.272	1.992	2.513	0	0.808	0.978	0.693

T/Equity	DTE GY Equity	DHL GY Equity	FME GY Equity	MBG GY Equity	IFX GY Equity	DB1 GY Equity	VNA GY Equity	P911 GY Equity	HNR1 GY Equity	CON GY Equity
0.5	0	7.709	3.534	10.268	1.39	2.64	6.386	0	3.239	4.084
1	0	3.844	1.762	5.12	0.693	1.317	3.184	2.527	1.615	2.036
2	4.199	3.815	1.813	4.622	0.571	1.292	3.388	2.73	0.938	1.975
3	7.428	3.755	1.831	4.396	0.529	1.279	3.46	2.768	0.712	1.93
4	8.51	3.703	1.829	4.247	0.507	1.267	3.459	2.764	0.598	1.906
5	8.842	3.633	1.822	4.118	0.496	1.257	3.433	2.746	0.53	1.879
7	8.546	3.496	1.799	3.905	0.479	1.24	3.348	2.691	0.453	1.832
10	7.535	3.296	1.754	3.634	0.465	1.213	3.191	2.589	0.393	1.771

Table B.2 Dividend Yield term structures for the German stocks in [%]. Reference Date: EOY 2024

ENR GY Equity	80	100	120
0.5	55.56	51.77	50.21
1	54.65	51.53	49.75
2	53.23	51.15	49.83
3	52.52	50.84	49.73
4	51.99	50.54	49.55
5	51.55	50.25	49.34
7	50.9	49.79	49
10	50.2	49.27	48.58

SY1 GY Equity	80	100	120
0.5	25.65	20.63	19.99
1	24.44	21.22	20.22
2	23.57	21.37	20.09
3	23.31	21.58	20.46
4	23.29	21.81	20.79
5	23.33	22	21.07
7	23.41	22.28	21.45
10	23.24	22.26	21.52

PAH3 GY Equity	80	100	120
0.5	32.13	26.7	25.27
1	29.91	25.69	24.1
2	33.44	26.95	23
3	31.32	26.09	22.73
4	30.15	25.65	22.69
5	29.25	25.21	22.51
7	28.04	24.61	22.28
10	26.68	23.77	21.76

MTX GY Equity	80	100	120
0.5	34	28.05	25.18
1	31.3	27.65	25.51
2	29.98	27.51	25.61
3	29.51	27.49	25.8
4	29.28	27.51	26
5	29.13	27.53	26.15
7	29	27.63	26.43
10	28.66	27.48	26.47

RHM GY Equity	80	100	120
0.5	39.35	36.78	36.38
1	37.71	36.13	35.81
2	36.83	35.73	35.41
3	36.37	35.57	35.31
4	36.1	35.47	35.25
5	35.93	35.4	35.21
7	35.7	35.3	35.15
10	35.49	35.2	35.09

DTG GY Equity	80	100	120
0.5	31.83	27.24	26.98
1	30.81	27.09	26.01
2	29.3	26.39	25.23
3	28.55	26.29	25.27
4	28.44	26.24	25.31
5	27.87	26.21	25.34
7	27.54	26.16	25.39
10	27.24	26.11	25.43

SHL GY Equity	80	100	120
0.5	27.23	22.79	21.96
1	26.16	23.31	21.76
2	25.24	23.27	21.96
3	24.96	23.32	22.16
4	24.83	23.38	22.32
5	24.77	23.45	22.46
7	24.72	23.57	22.68
10	24.73	23.73	22.94

ZAL GY Equity	80	100	120
0.5	43.34	39.92	39.52
1	42.93	40.61	39.61
2	41.66	40.26	39.55
3	41.26	40.2	39.61
4	41.09	40.21	39.7
5	41.05	40.29	39.83
7	41.04	40.44	40.06
10	40.72	40.23	39.91

QIA GY Equity	80	100	120
0.5	29.43	24.29	24.88
1	26.13	23.24	22.88
2	25.26	23.07	21.97
3	24.96	23.22	22.24
4	24.83	23.34	22.44
5	24.83	23.49	22.65
7	24.84	23.7	22.94
10	24.78	23.81	23.11

SRT3 GY Equity	80	100	120
0.5	48.6	44.82	42.28
1	47.48	44.51	42.16
2	46.49	44.33	42.59
3	45.96	44.16	42.7
4	45.57	43.99	42.7
5	45.25	43.82	42.64
7	44.71	43.47	42.45
10	44.01	42.94	42.06

BNR GY Equity	80	100	120
0.5	25.53	23.85	21.81
1	25.11	22.63	20.09
2	26.1	23.25	21.13
3	25.54	23.14	21.35
4	25.31	23.18	21.62
5	25.06	23.12	21.72
7	24.81	23.14	21.95
10	24.19	22.78	21.77

AIR GY Equity	80	100	120
0.5	38.385	38.385	38.385
1	38.385	38.385	38.385
2	38.385	38.385	38.385
3	38.385	38.385	38.385
4	38.385	38.385	38.385
5	38.385	38.385	38.385
7	38.385	38.385	38.385
10	38.385	38.385	38.385

ALV GY Equity	80	100	120
0.5	24.26	16.97	15.87
1	22.73	17.69	16.17
2	20.62	18.11	16.85
3	19.98	18.02	17.16
4	19.81	18.25	17.54
5	19.52	18.26	17.69
7	19.34	18.41	17.96
10	18.96	18.23	17.88

RWE GY Equity	80	100	120
0.5	27.95	24.4	24.92
1	27.2	25.07	24.7
2	26.05	24.98	24.45
3	26.19	25.12	24.58
4	26.03	25.13	24.64
5	25.84	25.06	24.63
7	25.6	24.94	24.57
10	25.32	24.83	24.58

BAYN GY Equity	80	100	120
0.5	40.02	38.24	38.54
1	39.77	38.19	38.28
2	38.32	37.47	37.36
3	36.77	35.94	35.84
4	35.78	35.03	34.87
5	34.88	34.17	33.99
7	33.61	33.02	32.82
10	33.04	33.23	31.9

BMW GY Equity	80	100	120
0.5	32.77	27.69	26.22
1	30.77	26.68	24.81
2	27.44	25.29	24.08
3	26.96	25.33	24.43
4	26.4	25.3	24.75
5	26.11	25.24	24.79
7	25.91	25.25	24.88
10	25.55	25.04	24.75

CBK GY Equity	80	100	120
0.5	37.86	31.97	29.46
1	37.2	32.44	29.65
2	35.35	32.82	31.07
3	34.32	32.45	31.2
4	33.91	32.39	31.37
5	33.54	32.26	31.41
7	33.13	32.16	31.49
10	32.58	31.78	31.23

DBK GY Equity	80	100	120
0.5	34.75	30.04	28.56
1	33.88	30.74	29.07
2	32.1	30.16	29.27
3	31.29	30.18	29.79
4	31.29	30.45	30.15
5	31.19	30.51	30.25
7	31.26	30.77	30.57
10	31.05	30.66	30.51

BAS GY Equity	80	100	120
0.5	30.65	25.61	23.92
1	28.67	24.56	22.54
2	25.45	23.35	22.04
3	25.05	23.28	22.22
4	24.76	23.39	22.57
5	24.49	23.36	22.68
7	24.27	23.37	22.83
10	23.87	23.13	22.69

HEN3 GY Equity	80	100	120
0.5	23	17.6	17.35
1	21.53	18.1	17.25
2	20.14	18.06	17.42
3	19.61	18.05	17.48
4	19.34	18.09	17.59
5	19.12	18.08	17.64
7	18.88	18.1	17.73
10	18.54	17.88	17.53

SIE GY Equity	80	100	120
0.5	29.63	25.31	25.18
1	29.42	25.62	24.7
2	27.93	25.84	24.59
3	27.35	25.53	24.63
4	26.95	25.45	24.62
5	26.65	25.39	24.64
7	26.27	25.32	24.73
10	25.83	25.09	24.64

VOW3 GY Equity	80	100	120
0.5	31.04	27.63	26.57
1	29.57	26.43	25.22
2	27.71	25.33	24.25
3	27.15	25.48	24.96
4	27.07	25.67	25.24
5	26.88	25.7	25.32
7	26.71	25.76	25.47
10	26.61	25.8	25.56

EOAN GY Equity	80	100	120
0.5	26.36	20.77	19.31
1	24.49	20.45	18.38
2	22.55	20.11	18.76
3	21.74	19.74	18.99
4	21.39	19.81	19.22
5	21.09	19.79	19.28
7	20.76	19.85	19.47
10	20.43	19.7	19.38

BEI GY Equity	80	100	120
0.5	24.79	19.48	18.56
1	23.02	19.4	18.07
2	21.97	19.19	18.03
3	21.25	19.09	18.05
4	20.91	19.14	18.2
5	20.7	19.18	18.34
7	20.48	19.27	18.54
10	20.13	19.14	18.49

HEI GY Equity	80	100	120
0.5	30.72	26.36	25.34
1	29.8	26.35	25.04
2	28.33	26.14	25.28
3	27.72	26.05	25.36
4	27.39	26.05	25.47
5	27.17	26.03	25.52
7	26.91	26.03	25.62
10	26.61	25.89	25.54

MUV2 GY Equity	80	100	120
0.5	26.42	21.83	21.09
1	24.62	21.59	20.3
2	22.9	20.42	19.24
3	22.41	20.41	19.64
4	22.12	20.58	19.87
5	21.86	20.61	20.02
7	21.73	20.73	20.2
10	21.33	20.54	20.12

FRE GY Equity	80	100	120
0.5	29.17	24.02	22.64
1	27.38	23.77	22.21
2	26.14	24.04	22.57
3	25.7	23.99	22.85
4	25.51	24.12	23.18
5	25.36	24.21	23.43
7	25.25	24.36	23.77
10	24.96	24.28	23.85

SAP GY Equity	80	100	120
0.5	30.5	25.74	25.08
1	30.2	26.16	25.09
2	28.19	25.8	24.85
3	27.34	25.59	24.72
4	26.93	25.46	24.7
5	26.65	25.37	24.67
7	26.24	25.21	24.62
10	25.84	25.03	24.59

MRK GY Equity	80	100	120
0.5	30.23	25.01	24.4
1	28.54	25.32	24.57
2	27.12	25.32	24.73
3	26.68	25.28	24.79
4	26.44	25.29	24.85
5	26.28	25.29	24.88
7	26.04	25.28	24.94
10	25.88	25.18	24.84

ADS GY Equity	80	100	120
0.5	33.75	28.92	27.95
1	32.31	29.1	27.73
2	30.34	28.23	26.98
3	29.74	27.97	27.26
4	29.58	28.05	27.32
5	29.44	28.11	27.42
7	29.15	28.15	27.58
10	28.95	28.19	27.76

DTE GY Equity	80	100	120
0.5	22.92	17.27	17.53
1	21.39	16.42	16.52
2	17.74	15.63	16.32
3	16.83	16.65	17.54
4	17.04	17.35	17.82
5	17.45	17.76	18
7	18	18.01	18.02
10	18.29	18.18	18.17

DHL GY Equity	80	100	120
0.5	27.95	22.66	21.66
1	26.76	22.47	20.75
2	24.37	21.87	20.34
3	23.73	21.74	20.45
4	23.35	21.66	20.52
5	23.11	21.6	20.57
7	22.78	21.51	20.63
10	22.49	21.43	20.67

FME GY Equity	80	100	120
0.5	34.24	30.5	28.77
1	32.56	29.62	27.95
2	31.08	27.86	25.96
3	30.05	27.12	25.25
4	29.68	27.03	25.27
5	29.42	27.05	25.45
7	29.26	27.26	25.87
10	28.29	26.55	25.36

MBG GY Equity	80	100	120
0.5	32.64	26.85	24.7
1	30.82	26.05	23.96
2	29.64	25.95	24.22
3	29.95	27.1	25.86
4	29.6	27.31	26.23
5	29.45	27.51	26.55
7	29.44	27.93	27.12
10	29.74	28.58	27.91

IFX GY Equity	80	100	120
0.5	38.57	35.08	33.58
1	37.48	34.71	33.4
2	36.87	34.44	32.82
3	35.62	34.7	34.25
4	35.91	34.59	33.74
5	35.59	34.47	33.74
7	35.26	34.36	33.74
10	34.77	33.99	33.44

DB1 GY Equity	80	100	120
0.5	24.63	19.02	17.4
1	22.9	18.87	17.49
2	21.65	19.02	17.78
3	31.1	19.06	17.98
4	20.9	19.17	18.2
5	20.79	19.25	18.35
7	20.7	19.4	18.59
10	20.41	19.28	18.54

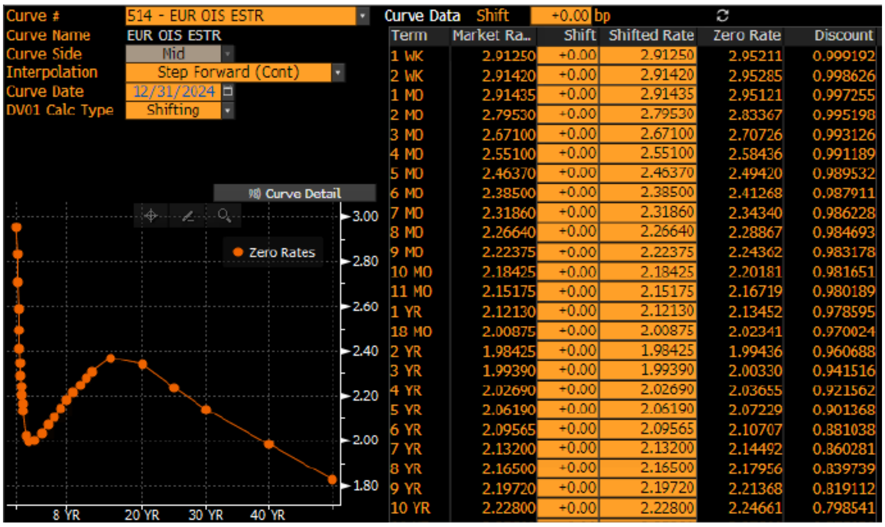
VNA GY Equity	80	100	120
0.5	31.72	29.26	28.07
1	31.46	28.91	27.75
2	30.48	28.66	27.12
3	30.37	28.83	27.56
4	30.36	29	27.9
5	30.34	29.1	28.11
7	30.37	29.29	28.45
10	30.14	29.22	28.5

P911 GY Equity	80	100	120
0.5	35.4	29.63	29.85
1	32.52	30.22	29.66
2	31.55	30.07	29.5
3	31.12	29.94	29.41
4	30.85	29.84	29.35
5	30.66	29.77	29.3
7	30.4	29.64	29.21
10	30.11	29.47	29.09

HNR1 GY Equity	80	100	120
0.5	25.58	20.24	19.45
1	24.62	20.63	19.4
2	23.65	20.91	19.7
3	23.4	21.16	20.02
4	23.34	21.4	20.32
5	23.38	21.63	20.6
7	23.57	22.07	21.13
10	23.98	22.71	21.86

CON GY Equity	80	100	120
0.5	34.91	30.2	28.29
1	34.04	30.67	28.74
2	31.05	28.87	27.35
3	30.77	29.02	27.74
4	30.68	29.18	28.05
5	30.59	29.24	28.22
7	30.52	29.38	28.5
10	30.31	29.34	28.58

Table B.3 Implied Volatility sections for 80% 100% 120% Moneyness. Volatility expressed in [%]. Reference Date: EOY 2024



Term	Market Ra..	Shift	Shifted Rate	Zero Rate	Discount
1 WK	2.91250	+0.00	2.91250	2.95211	0.999192
2 WK	2.91420	+0.00	2.91420	2.95285	0.998626
1 MO	2.91435	+0.00	2.91435	2.95121	0.997255
2 MO	2.79530	+0.00	2.79530	2.83367	0.995198
3 MO	2.67100	+0.00	2.67100	2.70726	0.993126
4 MO	2.55100	+0.00	2.55100	2.58436	0.991189
5 MO	2.46370	+0.00	2.46370	2.49420	0.989532
6 MO	2.38500	+0.00	2.38500	2.41268	0.987911
7 MO	2.31860	+0.00	2.31860	2.34340	0.986228
8 MO	2.26640	+0.00	2.26640	2.28867	0.984693
9 MO	2.22375	+0.00	2.22375	2.24362	0.983178
10 MO	2.18425	+0.00	2.18425	2.20181	0.981651
11 MO	2.15175	+0.00	2.15175	2.16719	0.980189
1 YR	2.12130	+0.00	2.12130	2.13452	0.978595
18 MO	2.00875	+0.00	2.00875	2.02341	0.970024
2 YR	1.98425	+0.00	1.98425	1.99436	0.960688
3 YR	1.99390	+0.00	1.99390	2.00330	0.941516
4 YR	2.02690	+0.00	2.02690	2.03655	0.921562
5 YR	2.06190	+0.00	2.06190	2.07229	0.901368
6 YR	2.09565	+0.00	2.09565	2.10707	0.881038
7 YR	2.13200	+0.00	2.13200	2.14492	0.860281
8 YR	2.16500	+0.00	2.16500	2.17956	0.839739
9 YR	2.19720	+0.00	2.19720	2.21368	0.819112
10 YR	2.22800	+0.00	2.22800	2.24661	0.798541

Figure B.1 Interest rates term structure – Tenor: 1 day (ESTER curve). Rates expressed in [%]. Reference Date: EOY 2024